University of Canterbury
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Aspects of Constructive Dynamical Systems

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Dedicated to the memory of my Grandmother
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Abstract

We give a Bishop-style constructive analysis of the statement that a continuous homomorphism from $\mathbf{R}$ onto a compact metric abelian group is periodic; constructive versions of this statement and its contrapositive are given. It is shown that the existence of a minimal period in general is not derivable, but the minimal period is derivable under a simple geometric condition when the group is contained in $\mathbf{R}^2$. A number of results about one-one and injective mappings are proved en route to our main theorems. A few Brouwerian examples show that some of our results are the best possible in a constructive framework.

Acknowledgements

Mathematical writing is fairly formulaic. The written structure of a proof is almost entirely fixed by its mathematical structure; the structure of a thesis is set by the structure of its mathematics. Because of this, the acknowledgements is perhaps the hardest part of a thesis to write: who to thank, how to apportion gratitude—it is almost impossible not to offend someone. To add to the problem there is not even a consensus as to the purpose of the acknowledgments.

Some say the point of acknowledgments is to drop names and bask in their reflected glory. In this manner, I would like to acknowledge a great debt of gratitude to my primary supervisor Douglas Bridges, without whom my thesis would have neither the quality of content nor the clarity of exposition (and would probably be in a far less interesting area of mathematics). I am also grateful to my second supervisor Iris Loeb and to the newly endoctored Hannes Diener for their periodic interest in and assistance toward the work of this thesis. To drop one last name, I thank Fred Richman for abstracting the problem this thesis addresses and for, in the process, illuminating the path towards a solution.

On the other hand, many believe the acknowledgements is a place to thank anybody who had a positive influence during the production of the work presented. Here, in addition to those previously mentioned, I thank the department at large, but in particular my fellow post grads, for providing a fun and friendly working environment (mostly) conducive to study. I thank my family and my friends outside the mathematics department for, among other things, their complete disinterest and ignorance towards mathematics in general; they provided a necessary counterbalance to my preoccupation. I also thank my father specifically for explaining the significance of the giant three toothed rat, and much other nonsensical input.

Finally, others still believe the purpose of the acknowledgments is solely to laud financial support; I must agree that I owe a lot to the University of Canterbury and the New Zealand Institute of Mathematics and its Applications for their generous support, and that this support deserves to be lauded. I would also like to thank Douglas and the department specifically for allowing me, as a mere Masters student, to attend the 'Mathematics, Algorithms and Proofs' conference in Italy, and in doing so giving me my first chance to explore continental Europe.

Christchurch, New Zealand, February 2009 Matthew Hendtlass

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Chapter 1

Introduction

In this section we give an introduction to constructive mathematics—in particular to Bishop's constructive mathematics (**BISH**), the system of mathematics in which this thesis is set—and introduce the problem discussed in this thesis. For a full development of **BISH** and other varieties of constructive mathematics see [6, 11, 12].

1.1 Constructive mathematics

Constructive mathematics diverges from classical (that is, traditional) mathematics in the assertion that an object exists only if it can be constructed. This simple deviance has a profound effect on the practice and scope of mathematics.

The explicit study of constructive mathematics originated with the work of Brouwer at the start of the 20th century. The major consequence of our strong interpretation of existence is a rejection of proof by contradiction ($\neg\neg P \Rightarrow P$), and a consequent refusal of the *law of excluded middle*

> **LEM**: For all syntactically correct statements P, either P is true or P is false.

from which it is derived. There are, however, many weaker non-constructive, or *omniscience*, principles prevalent in classical mathematics; Brouwer's first task was to identify these principles. Among the most important omniscience principles identified by Brouwer (though named by Bishop) are

- **The limited principle of omniscience (LPO)**: For any binary sequence $(a_n)_{n\geqslant 1}$, either $a_n = 0$ for all n or there exists n such that $a_n = 1$.
- **The weak limited principle of omniscience (WLPO)**: For any binary sequence $(a_n)_{n\geqslant 1}$, either $a_n = 0$ for all n or not $a_n = 0$ for all n.
- **The lesser limited principle of omniscience (LLPO)**: For any binary sequence $(a_n)_{n\geqslant 1}$ with at most one non-zero term, either $a_n = 0$ for all even n or $a_n = 0$ for all odd n.
- **Markov's Principle (MP)**: For any binary sequence $(a_n)_{n\geqslant 1}$, if it is impossible for $a_n = 0$ for all n, then there exists n such that $a_n = 1$.

The problem with **LEM** from a constructive viewpoint is that we cannot in general decide which of P or $\neg P$ is true, even though they are certainly not both false. More generally for Brouwer's mathematics to be algorithmic each step of a proof must be decidable. In particular, if we assert $A \vee B$ we must be able to either decide (that is, prove) that A is true or decide (prove) that B is true. As a consequence we reject the common classical definition (which, constructively, is equivalent to **LEM**)

$$A \vee B \Leftrightarrow \neg (\neg A \wedge \neg B) .$$

If **LPO** were true under this constructive interpretation of the disjunctive, then given any binary sequence $(a_n)_{n \geqslant 1}$, we could either decide that $a_n = 0$ for all n, or produce n such that $a_n = 1$. Since making such a decision, in general, requires us to check an infinite number of terms, nobody expects a constructive proof of **LPO** to exist, so we feel safe in rejecting it. Similar considerations result in the rejection of **WLPO** and **LLPO**. The non-constructive nature of Markov's principle, on the other hand, is a bit more debatable. Brouwer produced a philosophical argument showing **MP** and **LEM** to be equivalent; however, most mathematicians reject this example because it introduces temporal considerations into mathematics. Brouwer's argument can be formalised by assuming *Kripke's schema*:

> **KS:** For each proposition P there exists an increasing binary sequence $(a_n)_{n \geqslant 1}$ such that P holds if and only if $a_n = 1$ for some n.

Markov's principle + Kripke's schema is equivalent to **LEM**: considering any proposition P, let $\left(a_n^1\right)_{n \geqslant 1}$ and $\left(a_n^2\right)_{n \geqslant 1}$ be increasing binary sequences such that there exists n with $a_n^1 = 1$ if and only if P is true, and there exists n with $a_n^2 = 1$ if and only if P is false. Then defining $a_n = a_n^1 + a_n^2$, $(a_n)_{n \geqslant 1}$ is an increasing binary sequence. If $a_n = 0$ for each n, then neither P is true nor P is false giving the contradiction $\neg P \wedge \neg\neg P$. Hence by Markov's principle there exists $n > 0$ such that $a_n = 1$. Then either $a_n^1 = 1$ and P is true, or $a_n^2 = 1$ and P is false, so we have $P \vee \neg P$. Since **LEM** is false in Brouwer's intuitionistic mathematics, Markov's principle is refutable in intuitionistic mathematics + Kripke's schema. We, however, reject Markov's principle simply because it represents an unbounded search.

The second task of Brouwer was to identify the use of omniscience principles in classical mathematics to prove certain results, and to establish whether their use was necessary; that is, to establish whether the results constructively entail these principles. An example showing that a classically valid statement constructively entails an essentially nonconstructive principle is called a *Brouwerian counterexample*. It is important to note that a Brouwerian counterexample is not a genuine counterexample in the usual sense, but evidence that the statement under consideration does not permit a constructive proof. For example the statement

$$\neg(x = 0) \Leftrightarrow |x| > 0. \tag{1}$$

is constructively equivalent to **MP** and so is not constructively valid. To see this suppose (1) holds, and let $(a_n)_{n\geqslant 1}$ be a binary sequence such that it is impossible for all terms to be zero. Define

$$a = \sum_{n=1}^{\infty} 2^{-n} a_n.$$

Then $\neg(a = 0)$, so $|a| > 0$, by (1), and we can find $N > 0$ such that $|a| > 2^{-N}$. Checking the terms $a_1, a_2, \ldots, a_{N-1}$, we can find $n < N$ with $a_n = 1$. Hence (1) implies **MP**. This Brouwerian counterexample to (1) is said to be a *Brouwerian example of a real number x such that $\neg(x = 0)$ for which we cannot decide* $|x| > 0$. Classically we define $x \neq 0$ as $\neg(x = 0)$ and freely use equation (1); we have just shown that we cannot do this constructively. To resolve this problem we simply adopt the more powerful $x \neq 0 \Leftrightarrow |x| > 0$ as our definition of $x \neq 0$.

Another essentially nonconstructive statement is the *Law of Trichotomy*

$$\forall_{x\in\mathbf{R}} \left(x < 0 \vee x = 0 \vee x > 0\right),$$

which implies **LPO**: let $(a_n)_{n\geqslant 1}$ be a binary sequence and consider the number

$$a = \sum_{n=1}^{\infty} 2^{-n} a_n.$$

By the law of trichotomy, either $a = 0$ or $a > 0$. In the first case $a_n = 0$ for all n. In the second case pick $N > 0$ such that $a > 2^{-N}$. Then there must exist $n < N$ with $a_n = 1$.

Brouwer's final goal, often overlooked by the non-constructive mathematician, was to rebuild mathematics without the reliance on omniscience principles. For example, given that a proof of the law of trichotomy requires **LPO**, we might search for a similar, but necessarily weaker, result that is constructively valid. A constructive alternative to the law of trichotomy, sufficient for most applications, is provided by the *cotransitivity law*:

$$\forall_{x,y,z\in\mathbf{R}} \left(x < y \Rightarrow x < z \vee z < y\right).$$

Unfortunately, in pursuit of this final and most important goal Brouwer felt the need to adopt additional principles such as bar induction and his, classically false, continuity principle in order to prove any meaningful results. The bizarre justification of the former and bizarre consequences of the latter ensured that Brouwer's *intuitionistic* mathematics remained an obscure specialty oft derided by those classical mathematicians aware of its existence.

The next chapter in constructive mathematics, the Russian school of *recursive mathematics*, was authored by Markov et al. after the second world war. However, once again the practitioners of constructive mathematics felt the need to replace the omniscience principles they had rejected. In this case, recursive mathematicians adopted the Church-Markov-Turing thesis, and also accepted Markov's principle. Consequently, Russian recursive mathematics is essentially computability theory done with constructive logic augmented with **MP**. As with Brouwer's intuitionistic mathematics, the appeal of recursive mathematics was severely affected by a perceived incompatibility with classical mathematics.

Here then was the major hurdle that constructive mathematicians must overcome in order to gain the acceptance of the classical mathematician: to achieve sufficient mathematical depth without **LEM**, but also without having to assume any additional hypothesis (particularly classically false ones). Overcoming this obstacle seemed a near impossibility until, with the publication of his seminal monograph 'Foundations of Constructive analysis' [5] in 1967, Errett Bishop dispelled any concerns by developing much of modern analysis by purely constructive means. In addition, Bishop's work was free of the philosophical mysticism and the rigid formality that blighted the work of Brouwer and Markov respectively. With Bishop's constructive mathematics there was finally an appealing system of mathematics without the law of excluded middle.

1.2 Models of BISH

As we have said, the fundamental idea of constructive mathematics, in particular of Bishop's constructive mathematics, is that every proof is constructive, by which we mean that any proof contains an algorithm verifying the assertion it proves. This notion was made more precise with the following constructive interpretation of the logical connectives and quantifiers, called the *BHK-interpretation* after Brouwer, Heyting and Kolmogorov.

- $P \vee Q$: we have either a proof of P or a proof of Q.
- $P \wedge Q$: we have a proof of P and a proof of Q.
- $P \Rightarrow Q$: we can convert any proof of P into a proof of Q.
- $\neg P$: assuming P we can derive a contradiction.
- $\exists x P(x)$: we have an algorithm that computes a certain x and another that shows that $P(x)$ holds.
- $\forall x \in \mathcal{A} P(x)$: we have an algorithm which, applied to x and a proof that $x \in \mathcal{A}$, shows that $P(x)$ holds.

It turns out that the intuitionistic school of Brouwer (**INT**), and the Russian school of recursive mathematics (**RUSS**)[1] are based on this intuitionistic logic, and that **BISH** is precisely mathematics done with this logic.

[1]For a full development of intuitionistic and recursive mathematics see [11].

In Bishop's constructive mathematics, algorithm is considered to be a primitive notion interpreted, vaguely, as a 'finite routine'. Bishop's refusal to pin down the definition of an algorithm led to criticism, but this flexibility means that the results of **BISH** are immediately valid in both **INT** and **RUSS**, as well as classically, and in any other reasonable system of mathematics. In contrast, there are intuitionistic and recursive results which are classically false; conversely, there are classical results which are false in **INT** and **RUSS**. In particular, it can be shown (see [11]) that **LLPO** is false in both **INT** and **RUSS**—this provides further evidence for our rejection of **LLPO**, as well as the stronger principles of **WLPO**, **LPO** and **LEM**. In fact, we can characterise intuitionistic, recursive and classical mathematics in terms of **BISH** as follows; see Figure 1.

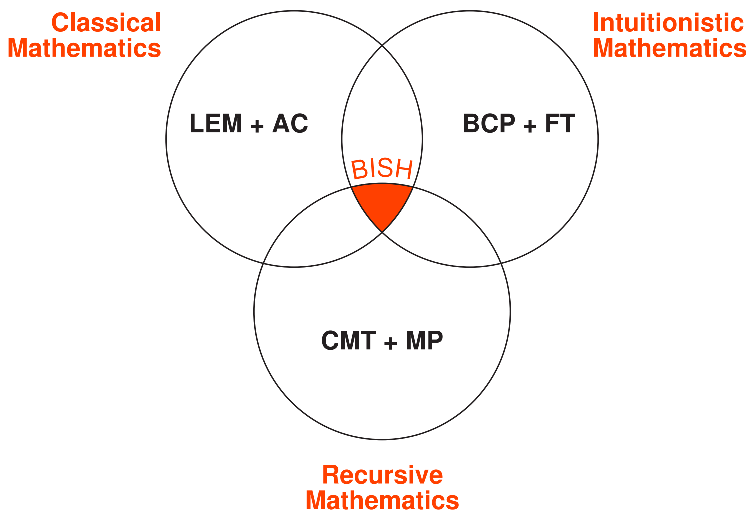

Figure 1: **BISH** the centre of modern mathematics, or at least at the centre of **INT**, **RUSS** and **CLASS**.

- **INT** is equivalent to **BISH** with the addition of Brouwer's Continuity Principle

  BCP : (1) Any function from $\mathbf{N}^{\mathbf{N}}$ to $\mathbf{N}$ is continuous.
  (2) If $P \subset \mathbf{N}^{\mathbf{N}} \times \mathbf{N},$ and for each $\mathbf{a} \in \mathbf{N}^{\mathbf{N}}$ there exists $n \in \mathbf{N}$ with $(\mathbf{a}, n) \in P,$ then there is a function $f : \mathbf{N}^{\mathbf{N}} \to \mathbf{N}$ such that $(\mathbf{a}, f(\mathbf{a})) \in P$ for all $a \in \mathbf{N}^{\mathbf{N}}$.

  and the full Fan Theorem (**FT**), which is equivalent (over **BISH**) to the Heine-Borel Theorem for general metric spaces [16].

- **RUSS** is equivalent to **BISH** with the Church-Markov-Turing Thesis (**CMT**) and Markov's Principle (**MP**). An important consequence of

CMT is that there is an enumeration of the set of partial functions from **N** to **N** with countable domains—this is known as **CPF**.

- **CLASS** is equivalent to **BISH** plus **LEM** and the Axiom of Choice

  AC: If X, Y are inhabited sets, S is a subset of $X \times Y$, and for each $x \in X$ there exists $y \in Y$ such that $(x, y) \in S$, then there exists a *choice function* $f : X \to Y$ such that $(x, f(x)) \in S$ for each $x \in X$.

Figure 1 is slightly misleading, in that there are results which hold in **INT**, **RUSS** and **CLASS** that are independent of **BISH**. An example is the boundedness principle[2] **BD-N** introduced by Ishihara in [23]. A subset A of **N** is *pseudobounded* if $\lim_{n\to\infty} n^{-1}a_n = 0$ for each sequence $(a_n)_{n\geqslant 1}$ in A. **BD-N** asserts that

BD-N: Every countable pseudobounded set is bounded.

The following figure summarises the interrelationship of the principles introduced so far[3], all of the implications are strict (see [2, 11, 23]).

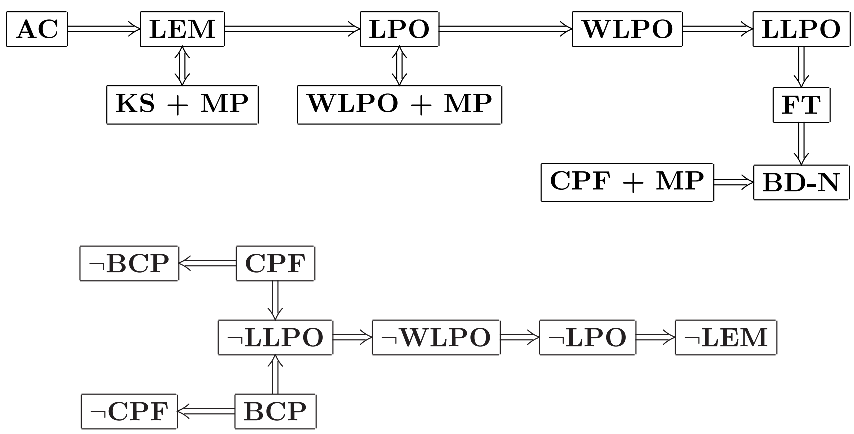

Figure 2: Summary of the interrelationship, within **BISH**, of various principles from **INT**, **RUSS** and **CLASS**.

In **BISH** we reject the axiom of choice because, in general, there is no way to construct the choice function of a set S; in fact, **AC** has been shown to imply the law of excluded middle [17, 20]. There are, however, choice axioms which are often (though not universally) accepted in constructive mathematics; namely, the *Principle of Countable Choice*, which states that every countable set has a choice function, and the stronger *Principle of Dependent Choice*:

[2]There is another model of **BISH** in which **BD-N** is provably false [26].
[3]To show that **CPF** implies ¬**LLPO** requires countable choice; see below.

> If X is a set, $a \in X$; S is a subset of $X \times X$; and for each $x \in X$ there exists $y \in X$ such that $(x, y) \in S$; then there exists a sequence $(x_n)_{n \geqslant 1}$ in X such that $x_1 = a$ and $(x_n, x_{n+1}) \in S$ for each positive integer n.

We finish our introduction to constructive mathematics by commenting on contraposition in **BISH**. The standard logical argument for taking the contrapositive is as follows. Given $A \Rightarrow B$, assume $\neg B$ and A. Then, by modus ponens, B holds, but this contradicts $\neg B$. Therefore if $\neg B$ holds, then we must have $\neg A$; that is, $\neg B \Rightarrow \neg A$. This simple argument is clearly consistent with our intuitionistic interpretation of the logical connectives and quantifiers, so contraposition is constructively valid. However, constructively negation statements are weak; and contrapositives are correspondingly weak. In particular, the contrapositive of $\neg B \Rightarrow \neg A$ ($\neg\neg A \Rightarrow \neg\neg B$) is not equivalent to $A \Rightarrow B$. Consequently, a result and its classical contrapositive typically split into two distinct results, requiring quite distinct proofs, when viewed in the constructive setting.

1.3 Some constructive definitions

As we have seen, when working constructively we need to choose our definitions with care; we strive to pick the 'best' (computationally most informative) definition from a set of classically equivalent ones. In this section we present some constructive definitions and some Brouwerian examples illustrating their distinction, when viewed constructively, from standard classical ones. Definitions that coincide with the classical ones are used without further comment.

A set S is said to be *inhabited* if there exists (that is, we can construct) an element of S. This is a stronger property than the classical notion of S being nonempty; the equivalence of these two notions implies **LEM**: consider the set

$$S = \{x : (x = 1) \wedge (P \vee \neg P)\},$$

where P is any syntactically correct statement.

There are also cases where we must introduce new definitions for notions implicit (and trivial) when working classically, but which do not hold universally in our constructive framework. Our next definition is of this type.

An inhabited subset S of a metric space X is said to be *located* if for each $x \in X$ the *distance*

$$\rho(x, S) = \inf \{\rho(x, s) : s \in S\}$$

exists. If the *metric complement* of S,

$$-S = \{x \in X : \rho(x, S) > 0\},$$

is also located, we then say that S is *bilocated*. The notion of locatedness plays a central role in constructive analysis, but it is often the case that a weaker condition will suffice. Let S be an inhabited subset of a metric space X and let $x \in X$. We write '$\rho(x, S) > 0$' as a shorthand for 'x is bounded away from S', without assuming that S is located. Likewise, we write '$\rho(x, S) = 0$' as a shorthand for

$$\forall_{\varepsilon>0} \exists_{s \in S} \left(\rho(0, s) < \varepsilon\right).$$

We then say that S is *weakly located at* x if for all $\varepsilon > 0$, either $\rho(x, S) > 0$ or $\rho(x, S) < \varepsilon$. If the *complement*

$$\sim S = \{x \in X : \forall s \in S(x \neq s)\}$$

of S is also weakly located at x, then S is said to be *weakly bilocated at* x. We say that S is *weakly (bi)located* if it is weakly (bi)located at x for each x in X. The statement 'every inhabited subset of $\mathbf{R}$ is weakly located at 0' is equivalent to **LEM**.

Let $\varepsilon > 0$. We call $\{x_1, \ldots, x_n\}$ an ε-*approximation* to S if each $x_i \in S$ and for each $s \in S$ there exists i $(i \leqslant i \leqslant n)$ such that $\rho\left(s, x_i\right) < \varepsilon$. We say that S is *totally bounded* if for each $\varepsilon > 0$ there exists an ε-approximation to S. Classically, boundedness and total boundedness are equivalent for finite dimensional spaces. The set $S = \{x : (x = 1 \wedge P)\} \cup \{0\}$ shows that constructively this is not the case. A metric space is then called *compact* if it is complete and totally bounded.

The following example shows that, even though the constructive and classical definitions coincide, there are sets which are complete classically, but not constructively. Define a function $\theta(t) : \mathbf{R} \to \mathbf{R}^2$ by

$$\theta(t) = \left(\sin\left(\frac{\pi t}{1 + |t|}\right), \sin\left(2\frac{\pi t}{1 + |t|}\right)\right).$$

Figure 3 shows $G \equiv \theta(\mathbf{R})$.

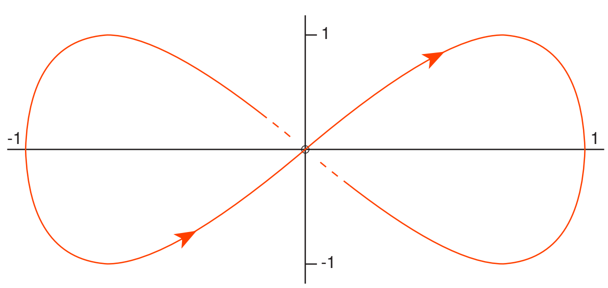

Figure 3: The image of the real line under θ.

We show that the completeness of G is equivalent to **LPO**. To this end, assume **LPO** and let $(t_n)_{n\geqslant 1}$ be a sequence in $\mathbf{R}$ such that $(\theta(t_n))_{n\geqslant 1}$ is Cauchy in G. Using **LPO** we can decide whether or not $(t_n)_{n\geqslant 1}$ is bounded: with **LPO**, construct an increasing binary sequence $(\lambda_n)_{n\geqslant 1}$ such that

$$\begin{array}{lll} \lambda_n = 0 & \Rightarrow & \text{there exists } k\,(|t_k| > n) \\ \lambda_n = 1 & \Rightarrow & \text{for all } k\,(|t_k| \leqslant n)\,. \end{array}$$

Applying **LPO** to this sequence, either $\lambda_n = 0$ for all n, in which case $(t_n)_{n\geqslant 1}$ is unbounded, or there exists N such that $\lambda_N = 1$, so $t_k \leqslant N$ for all k. If $(t_n)_{n\geqslant 1}$ is bounded by N, then $(\theta(t_n))_{n\geqslant 1}$ is a Cauchy, and therefore convergent, sequence in the complete (by Lemma 15) space $\theta[-N, N]$. On the other hand, if $(t_n)_{n\geqslant 1}$ is unbounded, then by **LPO** there exists a monotone subsequence $(t_{n_k})_{k\geqslant 1}$ of $(t_n)_{n\geqslant 1}$; without loss of generality $(t_{n_k})_{k\geqslant 1}$ is strictly increasing. Let $\varepsilon > 0$. By the continuity of the sine function at π and 2π, there exists $t_\varepsilon > 0$ such that

$$\max\left\{\sin\left(\frac{\pi t}{1+|t|}\right), \sin\left(\left(2\frac{\pi t}{1+|t|}\right)\right)\right\} < \varepsilon$$

for all $t > t_\varepsilon$. By **LPO**, there exists $\kappa > 0$ such that $t_{n_\kappa} > t_\varepsilon$. Then

$$\rho\left((0,0), \left(\sin\left(\frac{\pi t_{n_k}}{1+|t_{n_k}|}\right), \sin\left(\left(2\frac{\pi t_{n_k}}{1+|t_{n_k}|}\right)\right)\right)\right) < \varepsilon,$$

for all $k \geqslant \kappa$. Hence $(\theta(t_n))_{n\geqslant 1}$ converges. The converse follows from Lemma 4.

A metric space X is said to be *noncompact* if for each compact $K \subset X$, the metric complement

$$X - K = \{x \in G : \rho(x, K) > 0\}$$

is inhabited. With this definition, the statement

(*) Every complete inhabited set that is not compact is noncompact.

is equivalent to **LEM**. To see this, define

$$X = \{x : x \geqslant 1 \wedge P\} \cup \{0\} \cup \{x : x \leqslant -1 \wedge \neg P\}\,.$$

Then $0 \in X$, so X is inhabited. If X is compact, then it is bounded; whence there exists $N \in \mathbf{N}$ such that $X \subset (-N, N)$. Now if P holds, then $N \in X \subset (-N, N)$, a contradiction. It follows that $\neg P$ holds. Similarly we have $\neg\neg P$, giving the contradiction $\neg P \wedge \neg\neg P$. Thus X is not compact, and so by (*) X is noncompact. Given $\{0\}$ is compact, let $x \in X - \{0\}$. Either $x > -1$ or $x < 1$. In the first case we have that P holds, and in the second that $\neg P$ holds. Hence (*) implies **LEM**. Even the statement 'every bounded complete inhabited subset of $\mathbf{R}$ that is not compact is noncompact' is essentially nonconstructive.

Lastly, we have another property that holds trivially in classical mathematics: a function $f : X \to Y$ between two metric spaces (or, more generally, two sets with inequality relations) is said to be *strongly extensional* if

$$\forall_{x \in X} \forall_{x' \in X} \left(f(x) \neq_Y f\left(x'\right) \Rightarrow x \neq_X x' \right).$$

1.4 The problem

Let X be a metric space, and let ϕ be a *dynamical system* on X: that is, a continuous mapping of $\mathbf{R} \times X$ into X such that for each $x \in X$, $\phi(0, x) = x$ and

$$\phi\left(t + t', x\right) = \phi\left(t, \phi\left(t', x\right)\right) \quad \left(t, t' \in \mathbf{R}\right).$$

The original goal of this thesis was to give a complete constructive characterisation of the following well-known classical result from the theory of dynamical systems.

> **COP:** If the *orbit*
>
> $$\mathbf{R} \cdot x = \{\phi(t, x) : t \in \mathbf{R}\}$$
>
> of x in X is compact, then it is *periodic*, in the sense that there exists $\tau > 0$ with $\phi(\tau, x) = x$ [25].

The main results of this thesis, however, concern Theorem 1 (below), which is an abstraction[4] of **COP**. Before we introduce this we need a few more definitions. A *metric abelian group*[5] is an abelian group G equipped with a metric such that the mapping $(x, y) \rightsquigarrow y - x$ is pointwise continuous at $(0, 0) \in G \times G$, and uniformly continuous on compact subsets of $G \times G$. The mappings $x \rightsquigarrow -x$ and $(x, y) \rightsquigarrow x + y$ are then pointwise continuous throughout their domains, and uniformly continuous on compact subsets of their domains. Moreover, for each positive integer n, the mapping $x \rightsquigarrow nx$ is continuous at 0. Note that if G is *locally compact*—that is, every bounded subset of G is contained in a compact subset of G—then the pointwise continuity of the mapping $(x, y) \rightsquigarrow y - x$ is a consequence of its uniform continuity on compact sets. We say that a homomorphism θ of the abelian group $\mathbf{R}$ into a metric abelian group G is *continuous* if it is uniformly continuous on each compact (or, equivalently, on each bounded) subset of $\mathbf{R}$.

Theorem 1 *Let θ be a continuous homomorphism of $\mathbf{R}$ onto a compact (metric) abelian group G. Then there exists $\tau > 0$ such that $\theta(\tau) = 0$.*

We call τ a *period* of θ, and say that θ is *periodic*. To see that Theorem 1 is an abstraction of **COP**, let $\phi : \mathbf{R} \times X \to X$ be a dynamical system and let $x \in X$ be such that $\mathbf{R} \cdot x$ is compact. Then $G = \mathbf{R} \cdot x$ taken with the metric induced

[4] Due to Fred Richman.

[5] We use the standard additive notation for all abelian groups.

from X and with addition defined by $\phi(s,x)+\phi(t,x)=\phi(s+t,x)$, is a compact metric group with $0=\phi(0,x)=x$. Let $\theta:\mathbf{R}\to G$ be given by $\theta(t)=\phi(t,x)$. Then θ is clearly a homomorphism from $\mathbf{R}$ to G, and ϕ is periodic if and only if there exists $\tau>0$ such that $\theta(\tau)=0$.

In the next chapter we prove a number of useful results about one-one and injective mappings. In Chapter 3 we prove a constructive version of Theorem 1 before considering the classically vacuous problem of finding the minimal period. Chapter 4 then examines the (classical) contrapositive of Theorem 1:

Theorem 2 *Let θ be a continuous, one-one homomorphism of $\mathbf{R}$ onto a complete (metric) abelian group G. Then G is noncompact.*

In the final chapter we outline a few remaining questions. In several places we apply results not yet proved, for this we apologise and assure the reader that no circular logic is involved.

Although we know of no reference for Theorem 1 in the literature, we believe that the following argument, based on the standard classical one used to prove **COP** (see [25]), would be the natural one for the classical mathematician to use. Assume the hypothesis of Theorem 1 and set

$$T_n=\{\theta(t):t\in\mathbf{R},t>n\}.$$

Then the closed subsets $\overline{T_n}$ of G are compact. Since $T_1\subset T_2\subset\cdots$, the set

$$\omega(G)=\bigcap_{n\geqslant 1}\overline{T_n}$$

is nonempty by Cantor's intersection theorem. Moreover, $\omega(G)$ is invariant under the mappings $\phi^r:\theta(t)\mapsto\theta(t+r)$ $(r\in\mathbf{R})$. Pick $\xi=\theta(t_0)\in G$ and let $t\in\mathbf{R}$. Then, by the invariance of $\omega(G)$ under the mapping ϕ^{t-t_0},

$$\begin{aligned}\theta(t) &= \theta(t_0+t-t_0)\\ &= \phi^{t-t_0}(\xi) \qquad \in \omega(G).\end{aligned}$$

Hence $\omega(G)=G$, so T_n is dense in G for each n. On the other hand, by the continuity of θ, the sets $\theta[-n,n]$ are compact and hence closed. Since G is complete and equals the union of the sets $\theta[-n,n]$ $(n\geqslant 1)$, it follows from the Baire category theorem that the interior of $\theta[-N,N]$ is nonempty for some N. Since T_{N+1} is dense in G, we can find t_1,t_2 such that $|t_1|\leqslant N$, $t_2\geqslant N+1$, and $\theta(t_1)=\theta(t_2)$. Setting $\tau=t_2-t_1$, we see that $\tau>0$ and $\theta(\tau)=\theta(t_2)-\theta(t_1)=0$.

Where, then, are the constructive flaws in this argument? There are several, beginning with the compactness of $\overline{T}_n$: for that, it is not enough, constructively, that $\overline{T}_n$ be a closed subset of the compact space G; it must also be located in G

(see Chapter 2 of [12]). The next constructive flaw in the proof lies in its claim that $\bigcap\limits_{n\geqslant 1} \overline{T}_n$ is nonempty; actually, we would want the set to be inhabited. In **BISH**, the statement

> *Every descending sequence of compact subsets of a metric space has inhabited intersection.*

is equivalent[6] to **LLPO**. To see that this statement implies **LLPO**, let $(a_n)_{n\geqslant 1}$ be a binary sequence with at most one nonzero term and consider the sets

$$X_n = \begin{cases} [0,1] \cap X_{n-1} & \text{if } a_n = 0 \\ \{0\} & \text{if } a_n = 1 \text{ and } n \text{ is even} \\ \{1\} & \text{if } a_n = 1 \text{ and } n \text{ is odd.} \end{cases}$$

If x is in $\bigcap_{n\geqslant 1} X_n$, then either $x > 0$ or $x < 1$. In the first case $a_n = 0$ for all even n, and in the second $a_n = 0$ for all odd n. The converse is proved in [22].

A further flaw in the classical argument under scrutiny is the claim that $\theta\,[-n, n]$ is compact, a claim based on the preservation of compactness by uniformly continuous mappings: it is well-known that if, for each real number a, the image of $[-1, 1]$ under the mapping $x \rightsquigarrow ax$ is compact, then **LLPO** is derivable in **BISH**.

We are almost there with our criticism of the classical proof of Theorem 1. The final constructive problem arises in the application of Baire's category theorem: although the intersection of a sequence of dense open sets in a complete metric space is, as classically, dense, the classical contrapositive version of Baire's theorem—the version applied above—does not hold in **BISH** without some quite strong extra hypotheses; see Chapter 2 of [11] and also the recent papers [24, 28].

[6] Even with the additional hypothesis that there is at most one point in the intersection, the existence of such a point is equivalent, over **BISH**, to a version of Brouwer's fan theorem; see [4].

Chapter 2

Preliminary results

Here we present some preliminary results, many interesting in their own right, for use in later chapters.

2.1 One-one and injective mappings

We begin by clarifying the constructive distinction between 'one-one' and 'injective'. A mapping $f : X \to Y$ between metric spaces is said to be

- **one-one** if
$$\forall_{x,x' \in X} \left(f(x) = f(x') \Rightarrow x = x' \right);$$
- **injective**, or an **injection**, if
$$\forall_{x,x' \in X} (x \neq x' \Rightarrow f(x) \neq f(x')).$$

Although an injective mapping is one-one, the converse is easily seen to be equivalent to Markov's principle. Our first result, however, tells us that for certain homomorphisms these notions coincide. We require one more definition: a *length function* on an abelian group Γ is a mapping $x \rightsquigarrow \|x\|$ of Γ into the nonnegative real line such that for each x in Γ, $\|x\| \neq 0$ if and only if $x \neq 0$, and $\|nx\| = |n| \, \|x\|$ for each integer n.

Proposition 3 *Let Γ be an abelian group with a length function. Then every one-one homomorphism of Γ onto a complete metric abelian group is injective.*

The key to proving Proposition 3 is the following, somewhat odd, lemma. The oddity derives from the highly nonconstructive conclusion: namely, **LPO**.

Lemma 4 *Let f be a one-one mapping of a set T onto a complete metric space X, let h be a mapping of T into $\mathbf{R}$, and let $t \in T$. Let $(t_n)_{n \geq 1}$ be a sequence of elements of T such that $f(t_n) \to f(t)$ and $h(t_1) < h(t_2) < h(t_3) < \cdots \to \infty$ as $n \to \infty$. Then* ***LPO*** *holds.*

Proof. Let $(a_n)_{n \geqslant 1}$ be an increasing binary sequence. If $a_n = 0$, set $x_n = f(t)$; if $a_n = 1 - a_{n-1}$, set $x_k = f(t_n)$ for each $k \geqslant n$. Then $(x_n)_{n \geqslant 1}$ is a Cauchy sequence in X: for if $\varepsilon > 0$ and we compute ν such that $\rho\left(f(t_n), f(t)\right) < \varepsilon$ for all $n \geqslant \nu$, then $\rho\left(f(t_m), f(t_n)\right) < \varepsilon$ for all $m, n \geqslant \nu$. Since X is complete, the sequence $(x_n)_{n\geqslant 1}$ therefore converges to a limit $x_\infty \in X$. There exists a unique $t_\infty \in G$ such that $x_\infty = f(t_\infty)$. Compute a positive integer N such that $h(t_N) > h(t_\infty)$. Either $a_N = 0$ or $a_N = 1$. In the former case, suppose that $a_m = 1 - a_{m-1}$ for some $m > N$. Then $x_\infty = f(t_m)$, so, since f is one-one, $t_\infty = t_m$ and therefore $h(t_m) = h(t_\infty) < h(t_N)$, which is absurd since $m > N$. Thus if $a_N = 0$, then $a_n = 0$ for all $n \geqslant N$ and hence for all n. ■

We now have the **proof of Proposition 3**:

Proof. Let $x \rightsquigarrow \|x\|$ be the length function on Γ, and let θ be a one-one homomorphism of Γ onto a complete metric abelian group G. Consider any $t \in X$ such that $t \neq 0$. It will suffice to prove that $\theta(t) \neq 0$ in G: for in that case, if t, t' are distinct elements of X, we have

$$\theta(t) - \theta(t') = \theta(t - t') \neq 0,$$

so $\theta(t) \neq \theta(t')$. Construct, then, an increasing binary sequence $(\lambda_n)_{n\geqslant 1}$ such that for each n,

$$\begin{aligned} \lambda_n = 0 &\Rightarrow \rho\left(0, \theta(nt)\right) < 1/n, \\ \lambda_n = 1 &\Rightarrow \rho\left(0, \theta(nt)\right) > 1/\left(n+1\right). \end{aligned}$$

We may assume that $\lambda_1 = 0$. If $\lambda_n = 0$, set $y_n = 0 \in G$. If $\lambda_{n+1} = 1 - \lambda_n$, set $y_k = \theta(nt)$ for all $k \geqslant n$. Then

$$\rho\left(y_m, y_n\right) < 1/n \quad \left(m > n\right),$$

so $(y_n)_{n\geqslant 1}$ is a Cauchy sequence in G. Since G is complete and θ maps onto G, there exists $t_\infty \in \Gamma$ such that $y_n \to \theta(t_\infty)$ as $n \to \infty$. Pick a positive integer N such that $N\|t\| > \|t_\infty\|$. If $\lambda_N = 1$, then $N\theta(t) = \theta(Nt) \neq 0$, so $\theta(t) \neq 0$. On the other hand, if $\lambda_N = 0$, suppose that $\lambda_{m+1} = 1 - \lambda_m$ for some $m \geqslant N$. Then $\theta(t_\infty) = \theta(mt)$, so ($\theta$ being one-one) $t_\infty = mt$; whence $m\|t\| = \|t_\infty\| < N\|t\|$ and therefore $m < N$, a contradiction. Thus $\lambda_n = 0$ for all $n > N$ and therefore for all n; whence $\theta(nt) \to 0$ as $n \to \infty$. It follows from Lemma 4 that **LPO** holds. Since **LPO** implies Markov's principle, and since (θ being one-one) $\neg\left(\theta(t) = 0\right)$, we see that in this case also, $\theta(t) \neq 0$. ■

In particular, Proposition 3 allows us to assume in Theorem 2 that θ is injective. With this in mind we derive some general results about continuous injections.

Lemma 5 *Let f be an injection of a metric space X onto a complete metric space Y. Let $x_0 \in X$ and $S \subset X$ be such that $\rho\left(x_0, S\right) > 0$ and $f(S)$ is weakly located at $f\left(x_0\right)$. Then either $\rho\left(f(x_0), f(S)\right) > 0$ or $\rho\left(f(x_0), f(S)\right) = 0$.*

Proof. Write $y_0 = f(x_0)$ and construct an increasing binary sequence $(\lambda_n)_{n\geqslant 1}$ such that

$$\lambda_n = 0 \Rightarrow \rho\left(y_0, f(S)\right) < 2^{-n},$$
$$\lambda_n = 1 \Rightarrow \rho\left(y_0, f(S)\right) > 0.$$

We may assume that $\lambda_1 = 0$. If $\lambda_n = 0$, set $y_n = y_0$. If $\lambda_n = 1 - \lambda_{n-1}$, pick $s \in S$ such that $\rho\left(y_0, f(s)\right) < 2^{-n+1}$ and set $y_k = f(s)$ for each $k \geqslant n$. Then $(y_n)_{n\geqslant 1}$ is a Cauchy sequence in Y: indeed $\rho\left(y_m, y_n\right) < 2^{-n}$ whenever $m \geqslant n$. Since Y is complete, there exists $y_\infty \in Y$ such that $y_n \to y_\infty$ as $n \to \infty$. Write $x_\infty = f^{-1}(y_\infty)$. Either $x_\infty \neq x_0$ or $\rho\left(x_\infty, x_0\right) < \rho\left(x_0, S\right)$. In the first case, since f is injective, $y_\infty \neq y_0$, so there exists N such that $y_N \neq y_0$; then $\lambda_N = 1$ and therefore $\rho\left(y_0, f(S)\right) > 0$. In the case $\rho\left(x_\infty, x_0\right) < \rho\left(x_0, S\right)$, we have $\rho\left(x_\infty, S\right) > 0$. If $\lambda_n = 1-\lambda_{n-1}$ for some n, then $f(x_\infty) = y_\infty = y_n \in f(S)$, which is absurd, since $x_\infty \in X - S$ and f is injective. Hence in this case, $\lambda_n = 0$ for all n, and therefore $\rho(y_0, f(S)) = 0$. ∎

For the next proposition we need a lemma from [8].

Lemma 6 *If **LPO** holds, then every sequentially continuous mapping of a compact metric space into* $\mathbf{R}$ *is bounded.*

Proposition 7 *Let f be a sequentially continuous, injective mapping of a metric space X onto a complete metric space Y. Then f maps complete, located subsets of X onto complete subsets of Y.*

Proof. Let K be a complete, located subset of X, and $(x_n)_{n\geqslant 1}$ a sequence in K such that $(f(x_n))_{n\geqslant 1}$ is a Cauchy sequence in Y. Since Y is complete and f maps onto Y, there exists $x \in X$ such that $f(x_n) \to f(x)$ as $n \to \infty$. Suppose that $\rho\left(x, K\right) > 0$. Since f is injective, f^{-1} is strongly extensional. We can now apply Lemma 6.6.9 of [12] to show that **LPO** holds. For each $t \in K$, if $f(x) = f(t)$, then $x = t \in K$, a contradiction; so, by **LPO**, $f(x) \neq f(t)$. Thus the mapping

$$g : t \rightsquigarrow \frac{1}{\rho\left(f(x), f(t)\right)}$$

is well defined on K. It is easily seen to be sequentially continuous on K. It follows from Lemma 6 that there exists $M > 0$ such that $g(t) \leqslant M$, and therefore $\rho\left(f(x), f(t)\right) \geqslant 1/M$, for all $t \in K$. In particular, $\rho\left(f(x), f(x_n)\right) \geqslant 1/M$ for all n, which is absurd. This final contradiction shows that $\rho\left(x, K\right) = 0$. Hence x belongs to the closed set K, and therefore $f(x) \in f(K)$. ∎

Corollary 8 *Let f be a continuous injection of a compact metric space K into a metric space. Then $f(K)$ is compact.*

Proof. Since f is uniformly continuous on K, $f(K)$ is totally bounded. By Proposition 7, $f(K)$ is complete and hence compact. ∎

Corollary 9 *Let θ be a continuous, one-one homomorphism of* $\mathbf{R}$ *onto a complete metric abelian group G. Then $\theta(K)$ is compact for each compact $K \subset$* $\mathbf{R}$.

Proof. Since, by Proposition 3, θ is injective, the result follows from Corollary 8. ∎

The following result will not be put to use, but is sufficiently interesting for us to include it here.

Proposition 10 *Let f be a continuous injective mapping of a locally compact metric space X onto a complete metric space Y, and let K be a compact subset of X such that $X - K$ is inhabited. Then $Y - f(K) = f(X - K)$.*

Proof. The function f is uniformly continuous on K, so $f(K)$ is totally bounded; by Proposition 7, it is also complete, and therefore compact. Consider any $x \in X - K$. By [12] (Proposition 3.1.1), there exists $y \in f(K)$ such that if $f(x) \neq y$, then $f(x) \in Y - f(K)$. Let $x' = f^{-1}(y) \in K$. Then $x \neq x'$, so, since f is injective, $f(x) \neq y$ and therefore $f(x) \in Y - f(K)$. We conclude that $f(X - K) \subset Y - f(K)$.

For the reverse inclusion, consider any $y \in Y - f(K)$. Writing

$$\varepsilon = \rho\left(y, f(K)\right),$$

compute $\delta \in (0,1)$ such that if $x_i \in X$, $\rho\left(x_i, K\right) \leqslant 1 \quad (i = 1, 2)$, and $\rho\left(x_1, x_2\right) < \delta$, then $\rho\left(f(x_1), f(x_2)\right) < \varepsilon$. (This is the only place where we use the local compactness of X.) If $\rho\left(f^{-1}(y), K\right) < \delta$, then $\rho\left(y, f(K)\right) < \varepsilon$, a contradiction. Hence $\rho\left(f^{-1}(y), K\right) \geqslant \delta$. Thus $y \in f(X - K)$. ∎

2.2 Compactness, (co)locatedness and Baire's theorem

It is well-known that totally bounded subsets of X are located. Less well known, but not hard to prove, are that

(a) located subsets of a separable metric space are separable, and

(b) in **BISH + LPO**, separable subsets of X are located.

The proof of (a) is similar to that of Proposition 11 below. To prove (b), for a seperable subset A of X, let α, β be rational numbers with $\alpha < \beta$, and let $(a_n)_{n \geqslant 1}$ be a dense sequence in A. Fix $x \in X$ and construct a binary sequence $(\lambda_n)_{n \geqslant 1}$ such that

$$\begin{aligned} \lambda_n = 0 &\Rightarrow \rho\left(x, a_n\right) > \alpha \\ \lambda_n = 1 &\Rightarrow \rho\left(x, a_n\right) < \beta. \end{aligned}$$

Applying **LPO** to $(\lambda_n)_{n\geqslant 1}$, either $\lambda_n = 0$ for all n, in which case $\rho(x,a) \geqslant \alpha$ for all $a \in A$, or there exists n such that $\lambda_n = 1$, so that $\rho(x, a_n) < \beta$. It now follows from the constructive least upper bound principle (Theorem 2.2.18 of [12]) that $\rho(x, A) = \inf\{\rho(x,a) : a \in A\}$ exists.

Proposition 11 *Let Y be a located subset of a separable metric space X. Then $-Y$ is separable.*

Proof. Let $(x_n)_{n\geqslant 1}$ be a dense sequence in X. Construct a binary double sequence $(\lambda_{mn})_{m,n\geqslant 1}$ such that

$$\lambda_{mn} = 0 \Rightarrow \rho(x_m, Y) < 2/n,$$
$$\lambda_{mn} = 1 \Rightarrow \rho(x_m, Y) > 1/n.$$

Then

$$\{(m,n) : \lambda_{mn} = 1\}$$

is a countable set; so

$$S = \{x_m : \exists_n (\lambda_{mn} = 1)\}$$

is a countable subset of $-Y$. Given $x \in -Y$ and $\varepsilon > 0$, pick a positive integer $n > 1/\varepsilon$ such that $\rho(x,Y) > 3/n$. There exists m such that $\rho(x, x_m) < 1/n < \varepsilon$. Then $\rho(x_m, Y) > 2/n$, so $\lambda_{mn} \neq 0$ and therefore $\lambda_{mn} = 1$. Hence $x_m \in -Y$. It follows that S is a countable dense subset of $-Y$. ■

Proposition 12 *The following are equivalent over **BISH**.*

(i) ***LPO***.

(ii) *Every located subset of a separable metric space is bilocated.*

Proof. The implication (i) $\Rightarrow$ (ii) follows from Proposition 11 and (b) of the paragraph preceding it. Conversely, assume (ii) and consider any increasing binary sequence $(a_n)_{n\geqslant 1}$. Define

$$S = \left\{ \frac{1-a_n}{n} : n \geqslant 1 \right\},$$

and let X be the discrete metric space $\{0,2\} \cup S$. Then X is separable, S is located in X, and so $-S$ is located in X. Either $\rho(0, -S) < 2$ or $\rho(0, -S) > 0$. In the first case, picking $y \in -S$ with $|y| < 2$, we see that $\neg (y \neq 0)$; so $y = 0 \in -S$, and we can compute N such that $\rho(0, S) > 1/N$; then $a_N = 1$. In the case $\rho(0, -S) > 0$, we must have $a_n = 0$ for all n. ■

Recall that the form of Baire's (category) theorem whose classical proof carries over to the constructive setting is the one stating that the intersection of a sequence of dense open subsets of a complete metric space is itself dense. However, its classical contrapositive does not hold in **BISH** without some additional hypotheses. For later use, we prove a constructive version of that contrapositive.[7]

Theorem 13 *Let $(C_n)_{n\geqslant 1}$ be a sequence of closed bilocated subsets of a complete metric space X such that $X = \bigcup_{n\geqslant 1} C_n$. Then there exists n such that C_n° (the interior of C_n) is inhabited.*

Proof. For each n write

$$U_n = -C_n \cup \{x \in X : \exists_k \exists_y (y \in C_k^\circ)\}.$$

Then U_n is certainly open. To see that it is dense in X, let $x \in X$ and $\varepsilon > 0$. Either $\rho(x, -C_n) < \varepsilon$ and therefore there exists $y \in -C_n \subset U_n$ such that $\rho(x, y) < \varepsilon$; or else $r = \rho(x, -C_n) > 0$. In the latter case, consider any $y \in B(x, r)$. If $\rho(y, C_n) > 0$, then $y \in -C_n$ and therefore $\rho(x, -C_n) < r$, a contradiction. Hence $\rho(y, C_n) = 0$, so $y \in \overline{C_n} = C_n$. Hence $B(x, r) \subset C_n$, so $x \in C_n^\circ$ and $U_n = X$.

We can now apply the usual form of Baire's theorem ([5], page 87, Theorem 4; [6], page 93, Theorem (3.9)) to construct $\xi \in \bigcap_{n\geqslant 1} U_n$. Pick N such that $\xi \in C_N$. Then $\xi \notin -C_N$, so

$$\xi \in \{x \in X : \exists_k \exists_y (y \in C_k^\circ)\}$$

and therefore C_k° is inhabited for some k. ■

Corollary 14 *Assume **LPO**, and let $(C_n)_{n\geqslant 1}$ be a sequence of closed, located subsets of a complete, separable metric space X such that $X = \bigcup_{n\geqslant 1} C_n$. Then there exists n such that C_n° (the interior of C_n) is inhabited.*

Proof. By Proposition 12, each C_n is bilocated; so the result follows from Theorem 13. ■

Proposition 15 *If **LLPO** holds, then the image of a compact metric space under a uniformly continuous mapping is complete.*

Proof. Assume **LLPO** and let f be a uniformly continuous mapping of a compact space X into a metric space Y. Then $f(X)$ is a totally bounded subset of Y; so its closure in the completion $\widehat{Y}$ of Y is compact. Let $(x_n)_{n\geqslant 1}$ be a sequence in X such that $(f(x_n))_{n\geqslant 1}$ is a Cauchy, and therefore convergent, sequence in $\widehat{Y}$. Let $f(x_n) \to y$ as $n \to \infty$. Define a uniformly continuous mapping $g : X \to \mathbf{R}$ by

$$g(x) = \rho(y, f(x)).$$

By **LLPO**, there exists $\xi \in X$ such that $g(\xi) = \inf g = 0$ (see [22]). Thus $y = f(\xi) \in f(X)$. Hence $f(X)$ is complete and therefore compact. ■

[7] Our version is stated, without a full proof, as Theorem (2.5) in Chapter 2 of [11]; our proof is neater than the one suggested in that reference. Fred Richman [28] has shown recently how both the 'dense open sets' version of Baire's theorem and a strong constructive form of the 'closed sets' version can be produced as consequences of a single lemma.

We now introduce a collection of sets that, as well as allowing us to prove the converse of Proposition 15, provides us with some important Brouwerian examples in the later chapters. For $a \in \mathbf{R}$, define

$$G_a = \left\{\theta(t) = \left(e^{2\pi i t}, a e^{\pi i t}\right) : t \in \mathbf{R}\right\} \subset S^1 \times \mathbf{C}, \tag{2}$$

where

$$S^1 = \{z \in \mathbf{C} : |z| = 1\}.$$

Then

$$\left(e^{2\pi i t}, a e^{\pi i t}\right) + \left(e^{2\pi i t'}, a e^{\pi i t'}\right) = \left(e^{2\pi i (t+t')}, a e^{\pi i (t+t')}\right)$$

defines an addition operation that turns G_a into an abelian group with identity $0 \equiv (1, a)$, and

$$\theta_a(t) = \left(e^{2\pi i t}, a e^{\pi i t}\right) \tag{3}$$

is a continuous homomorphism of $\mathbf{R}$ onto G_a. Moreover, $\theta_a(2) = 0$. Since we are concerned with homorphisms onto complete metric abelian groups, the next lemma proves to be very useful.

Lemma 16 *The following are equivalent.*

(i) *The set G_a is complete for each $a \in \mathbf{R}$.*

(ii) ***LLPO**.*

Proof. Assuming (i), let $(a_n)_{n \geqslant 1}$ be a binary sequence with at most one nonzero term, and define

$$a = \sum_{n=0}^{\infty} 2^{-n} a_n.$$

If $a_k = 0$ for all $k \leqslant n$, set $t_n = n$; if $a_n = 1$, set $t_j = n$ for all $j \geqslant n$. For all m, n we have

$$\rho\left(\theta_a(t_m), \theta_a(t_n)\right) = a\left|e^{\pi i t_m} - e^{\pi i t_n}\right| \leqslant 2a.$$

If $\varepsilon > 0$, then either $2a < \varepsilon$, in which case $\rho\left(\theta_a\left(t_n\right), \theta_a\left(t_m\right)\right) < \varepsilon$ for all m and n, or else $a > 0$. In the latter case, pick a positive integer N such that $2^{-N} < a$; then $a = 2^{-\nu}$ for some $\nu < N$, so $t_m = t_n = \nu$, and therefore $\rho\left(\theta_a(t_m), \theta_a(t_n)\right) = 0 < \varepsilon$, for all $m, n \geqslant \nu$. Since $\varepsilon > 0$ is arbitrary, it follows that $(\theta_a(t_n))_{n \geqslant 1}$ is a Cauchy sequence in G_a. Using (i), we can find $t \in \mathbf{R}$ such that $\theta_a(t_n) \to \theta_a(t)$ as $n \to \infty$. Choose a positive integer κ such that $\kappa \leqslant t < \kappa + 2$. Either $t > \kappa$ or $t < \kappa + 1$. In the first case we must have $a_k = 0$ for all $k \leqslant \kappa$ and all $k > \kappa + 1$; so either $a_k = 0$ for all k or $a_{\kappa+1} = 1$; in either event we have $a_k = 0$ for all k congruent to κ. In the case $t < \kappa + 1$, a similar argument shows that $a_k = 0$ for all k congruent to $\kappa + 1$. We now conclude that (i) implies **LLPO**.

Conversely, assuming (ii), fix a in $\mathbf{R}$ and let $(t_n)_{n \geqslant 1}$ be a sequence in $\mathbf{R}$ such that $(\theta_a(t_n))_{n \geqslant 1}$ is a Cauchy sequence in G_a and hence in $S^1 \times \mathbf{C}$; then $\left(e^{2\pi i t_n}\right)_{n \geqslant 1}$

is a Cauchy sequence in S^1. Since S^1 and $S^1 \times \mathbf{C}$ are complete spaces, there exist $t \in \mathbf{R}$ and $z = (z_1, z_2) \in S^1 \times \mathbf{C}$ such that

$$\max\left\{\left|e^{2\pi i t_n} - e^{2\pi i t}\right|, \rho\left(\theta_a(t_n), z\right)\right\} \to 0 \text{ as } n \to \infty.$$

By **LLPO**, either $\rho\left(z, \theta_a(t)\right) \leqslant \rho\left(z, \theta_a(t+1)\right)$, in which case we take $s = t$, or else

$$\rho\left(z, \theta_a(t+1)\right) \leqslant \rho\left(z, \theta_a(t)\right) = \rho\left(z, \theta_a(t+2)\right),$$

when we take $s = t+1$. Thus in either case,

$$\rho\left(z, \theta_a(s)\right) \leqslant \rho\left(z, \theta_a(s+1)\right). \tag{4}$$

Note that

(*) if $s = t$, then $e^{\pi i(s+1)} = -e^{\pi i t} = -e^{\pi i s}$; and if $s = t+1$, then $e^{\pi i s} = -e^{\pi i t}$ and $e^{\pi i(s+1)} = e^{\pi i(t+2)} = e^{\pi i t}$.

We show that $z = \theta_a(s) \in G_a$. To that end, suppose that $z \neq \theta_a(s)$. Since

$$e^{2\pi i s} = e^{2\pi i t} = \lim_{n\to\infty} e^{2\pi i t_n} = z_1,$$

we see that

$$a e^{\pi i s} \neq z_2 = \lim_{n\to\infty} a e^{\pi i t_n}$$

and therefore that $a \neq 0$. Compute a positive integer N such that for all $n \geqslant N$,

$$\rho\left(\theta_a\left(t_n\right), z\right) < \delta = \frac{1}{2}\min\left\{|a|, |a|^2, \rho(\theta_a(s), z)\right\},$$

and

$$\left|e^{2\pi i t_n} - e^{2\pi i t}\right| < \left(\frac{\delta}{a}\right)^2.$$

Noting (*), we see that for such n,

$$\begin{aligned}\left|e^{\pi i t_n} - e^{\pi i s}\right|\left|e^{\pi i t_n} - e^{\pi i(s+1)}\right| &= \left|e^{\pi i t_n} - e^{\pi i t}\right|\left|e^{\pi i t_n} + e^{\pi i t}\right| \\ &= \left|e^{2\pi i t_n} - e^{2\pi i t}\right| < \left(\frac{\delta}{a}\right)^2.\end{aligned} \tag{5}$$

If $\left|e^{\pi i t_n} - e^{\pi i s}\right| < \delta/|a|$, then (note that $(\delta/a)^2 < \delta$)

$$\begin{aligned}\rho\left(\theta_a(s), z\right) &\leqslant \rho(\theta_a(t_n), \theta_a(s)) + \rho\left(\theta_a(t_n), z\right) \\ &< \max\left\{\left(\frac{\delta}{a}\right)^2, |a|\left|e^{\pi i t_n} - e^{\pi i s}\right|\right\} + \delta \\ &< 2\delta \leqslant \rho\left(\theta_a(s), z\right),\end{aligned}$$

which is absurd. Hence, by (5), $\left|ae^{\pi i t_n} - ae^{\pi i(s+1)}\right| < \delta$. Since $e^{2\pi i(s+1)} = e^{2\pi i s} = e^{2\pi i t}$, it follows that

$$\rho\left(\theta_a(t_n), \theta_a(s+1)\right) < \delta \leqslant \frac{1}{2}\rho(\theta_a(s), z)$$

for all $n \geqslant N$. Letting $n \to \infty$, we obtain $\rho\left(z, \theta_a(s+1)\right) \leqslant \frac{1}{2}\rho(\theta_a(s), z)$. It follows from this and (4) that $\rho\left(\theta_a(s), z\right) = 0$ and therefore $z = \theta_a(s)$, a contradiction. We now conclude that z does, after all, equal $\theta_a(s)$, and therefore that G_a is complete. Thus **LLPO** implies (i). ■

Corollary 17 *The following are equivalent.*

(1) ***LLPO****.*

(2) The image of a compact metric space under a uniformly continuous mapping is complete.

Proof. Given Proposition 15, all that remains is to show that *(2)* $\Rightarrow$ *(1).* Recall the abelian group G_a and the homomorphism $\theta : \mathbf{R} \to G_a$ from above. If (2) holds, then $G_a = \theta[0, 4]$ (Proposition 20) is complete and, being totally bounded, is therefore compact. It then follows from Lemma 16 that **LLPO** holds. ■

Chapter 3

Continuous homomorphisms of R onto a compact group

In this chapter we prove a constructive version of Theorem 1 and show it to be classically equivalent to that theorem. We then consider the classically vacuous problem of finding the minimal period of a periodic homomorphism.

3.1 Periodicity

The main result of this chapter is

Theorem 18 *Let θ be a continuous homomorphism of* $\mathbf{R}$ *onto a compact (metric) abelian group G such that $\theta(0,\infty)$ is open in G. Then there exists $\tau > 0$ such that $\theta(\tau) = 0$.*

If θ is a homomorphism of the additive abelian group $\mathbf{R}$ onto a metric group G, then for each $r \in \mathbf{R}$ we write $T_r = \theta(r, \infty)$. Before we prove Theorem 18, we need the following simple lemma.

Lemma 19 *Let θ be a continuous homomorphism from* $\mathbf{R}$ *onto a complete metric group G such that T_0 is open. Then T_r is open for all $r \in \mathbf{R}$.*

Proof. Fix $r \in \mathbf{R}$, and let $t > r$. Then $t - r > 0$, so, since T_0 is open, there exists $\varepsilon > 0$ such that $B(\theta(t-r), \varepsilon) \subset T_0$. By [6] (page 400, Proposition (1.2)), there exists $\delta > 0$ such that for all $s \in \mathbf{R}$, if $\rho(\theta(t), \theta(s)) < \delta$, then $\rho(\theta(t-r), \theta(s-r)) < \varepsilon$. Let $\theta(s) \in B(\theta(t), \delta)$; then

$$\theta(s-r) \in B(\theta(t-n), \varepsilon).$$

Hence there exists $t' > 0$ with $\theta(t') = \theta(s-r)$, so $\theta(s) = \theta(t'+r) \in T_r$ and $B(\theta(t), \delta) \subset T_r$. ■

Here is the **proof of Theorem** 18:

Proof. We prove first that, for each r, $\rho(0, T_r) = 0$. To do so, given $\varepsilon > 0$, compute $\delta > 0$ such that if $y, y' \in G$ and $\rho(y, y') < \delta$, then $\rho(0, y - y') < \varepsilon$; this is possible in view of [6] (page 400, Proposition (1.2)). Pick $t_1, \ldots, t_m$ in $\mathbf{R}$ such that $\{\theta(t_1), \ldots, \theta(t_m)\}$ is a δ-approximation to G, and let

$$t > \max\{t_1, \ldots, t_m\} + r.$$

There exists $i \leqslant m$ such that $\rho(\theta(t), \theta(t_i)) < \delta$ and therefore $\rho(0, \theta(t - t_i)) < \varepsilon$; moreover, $t - t_i > r$. Since $\varepsilon > 0$ is arbitrary, we conclude that $\rho(0, T_r) = 0$.

Now consider any $t \in \mathbf{R}$ and $\varepsilon > 0$. By [6] (page 400, Proposition (1.2)), there exists $\delta > 0$ such that if $y, y' \in G$ and $\rho(0, y - y') < \delta$, then $\rho(y, y') < \varepsilon$. The first part of the proof enables us to construct t' such that $t' > r + t$ and $\rho(0, \theta(t')) < \delta$. Then $\rho(\theta(t), \theta(t' - t)) < \varepsilon$, where $t' - t > r$. Since $t \in \mathbf{R}$ and $\varepsilon > 0$ are arbitrary, we conclude that T_r is dense in G. Hence $(T_n)_{n \geqslant 1}$ is a sequence of dense, open (by Lemma 19), subsets of G. Applying Baire's theorem, we see that $\bigcap_{n \geqslant 1} T_n$ is dense in G and so, in particular, contains $\theta(t_0)$ for some $t_0 \in \mathbf{R}$. Pick a positive integer $N > t_0$. Then $\theta(t_0) \in T_N$, so there exists t_0' such that $t_0' > N$ and $\theta(t_0) = \theta(t_0')$. Then $t_0' - t_0 > 0$ and $\theta(t_0 - t_0') = 0$. ∎

How reasonable is our assumption that $\theta(0, \infty)$ is open? First, we observe that if θ is periodic, then for each $t \in \mathbf{R}$ there exists $t' > 0$ such that $\theta(t) = \theta(t')$, so $T_0 = G$ is open. Now let θ be a continuous homomorphism of $\mathbf{R}$ onto any metric abelian group, and assume T_0 is not open. Let $\theta(t) = 0$; if $t \neq 0$, then θ is periodic, so T_0 is open, a contradiction. Hence $\neg(t \neq 0)$ and therefore $t = 0$. This shows that θ is one-one. On the other hand, we see from the first part[8] of the proof of Theorem 18 that

$$\{\theta(t) : |t| \geqslant 1\}$$

is dense and therefore located in G. It follows from Proposition 43 that θ^{-1} is uniformly continuous on G; whence T_0 is open in G. This contradiction shows, constructively, that T_0 cannot fail to be open; classically, it follows that T_0 is open.

In trying to prove that $\theta(0, \infty)$ is open a natural approach might be to show

(*) for all $t, \varepsilon > 0$ there exist $\delta > 0$ such that $B(0, \delta) \subset \theta(t - \varepsilon, t + \varepsilon)$.

The following Brouwerian example shows that this route is not open to us. Recall the sets G_a introduced in (2). We show that (*) implies that **LLPO** and **LPO** are equivalent. Assuming **LLPO**, G_a is compact by Lemma 16; for simplicity we shift our problem and consider $\theta_a(-1, \infty)$ and $t = 0$. Assume there exists $\delta > 0$ such that $B(0, \delta) \subset \theta_a(-1, 1)$. Then either $\rho(0, \theta_a(1)) > 0$, in which case $a > 0$, or $\rho(0, \theta_a(1)) < \delta$. In the latter case there exists $s \in (-1, 1)$ with $\theta_a(1) = (1, -a) = \theta_a(t)$. Clearly $s = 0$, so $(1, -a) = \theta_a(0) = (1, a)$ and $a = 0$. Hence (*) implies that $\forall_{x \in \mathbf{R}} (x = 0 \vee x \neq 0)$, which in turn implies **LPO**.

There are other reasonable assumptions we can add to Theorem 1 to get a constructive proof—it may be that no additional assumptions are necessary. If there exists a sequence of real numbers $(R_n)_{n \geqslant 1}$ strictly increasing to infinity

[8] This part does not use the hypothesis that T_0 is open.

such that $\theta\left[-R_n, R_n\right]$ is weakly bilocated for each n, then we can prove Theorem 1, without the hypothesis that $\theta(0,\infty)$ is open, by using a stronger version of Theorem 13 (see [28]). The existence of such a sequence is trivial both classically, where every set is located, and if θ is periodic where we set R_1 greater than the period. In particular, if there exists $t>0$ such that $G=\theta[0,t]$, then setting $R_n=nt$, say, we have that θ is periodic.

Next we consider what may initially appear to be a ridiculously trivial question: if a continuous homomorphism of $\mathbf{R}$ onto a metric abelian group is periodic, is the group compact?

Proposition 20 *Let θ be a continuous homomorphism of $\mathbf{R}$ onto a metric abelian group G such that there exists $\tau>0$ with $\theta(\tau)=0$. Then $\theta[0,\tau]$ is dense in G, which is totally bounded, and $G=\theta[0,2\tau]$.*

Proof. Given $\varepsilon>0$, compute $\delta\in(0,\tau)$ such that if $t,t'\in[-\tau,\tau]$ and $|t-t'|<\delta$, then $\rho\left(\theta(t),\theta(t')\right)<\varepsilon$. Fix $t\in\mathbf{R}$. Either $|t|<\delta$, in which case $\rho\left(0,\theta(t)\right)<\varepsilon$, or else $t\neq 0$. In the latter case, taking $t>0$ for illustration, we compute an integer $n\geqslant 0$ such that $n\tau\leqslant t<(n+2)\,\tau$. Then either $|t-(n+1)\,\tau|<\delta<\tau$ or $t\neq(n+1)\,\tau$. In the first case,

$$\rho\left(0,\theta(t)\right)=\rho\left(0,\theta(t)-(n+1)\,\theta(\tau)\right)=\rho\left(0,\theta\left(t-(n+1)\,\tau\right)\right)<\varepsilon.$$

In the second case, either $t\in[n\tau,(n+1)\,\tau)$ or $t\in((n+1)\,\tau,(n+2)\,\tau]$, so there exists $\nu\in\{n,n+1\}$ such that $t-\nu\tau\in[0,\tau]$; since

$$\rho\left(0,\theta(t)\right)=\rho\left(0,\theta(t-\nu\tau)+\nu\theta(\tau)\right)=\rho\left(0,\theta(t-\nu\tau)\right),$$

we have $\theta(t)\in\theta[0,\tau]$. Since ε and t are arbitrary, $\theta[0,\tau]$ is dense in G. The uniform continuity of θ on $[0,\tau]$ ensures that $\theta\,[0,\tau]$, and hence G, is totally bounded. For the final part of the proposition let $t\in\mathbf{R}$ and find $N\in\mathbf{N}$ such that $t\in[N\tau,(N+2)\tau]$. Then $t-N\tau\in[0,2\tau]$, so $\theta(t)=\theta(t-N\tau)\in\theta[0,2\tau]$. ■

Classically, under the hypotheses of Proposition 20 we would conclude that G is compact. Corollary 17 shows we cannot do so constructively.

We finish this section with a simple corollary of Theorem 18 relevant to the material of the next chapter.

Corollary 21 *Let G be a locally compact abelian group that is not compact, and let θ be a continuous homomorphism of $\mathbf{R}$ onto G. Then θ is injective.*

Proof. In view of the Proposition 3, it is enough to prove that θ is one-one. Accordingly, given $t\in\mathbf{R}$ such that $\theta(t)=0$, suppose that $t\neq 0$; in order to derive a contradiction, we may assume that $t>0$. Then, by the preceding proposition, G is totally bounded. Being complete, G is therefore compact—a contradiction. We conclude that $\neg\,(t\neq 0)$, from which it follows that $t=0$. Hence θ is one-one. It just remains to invoke Proposition 3. ■

3.2 Finding the minimal period

The next major question is: under the hypotheses of Theorem 18, when θ is nontrivial (that is, there exists $t > 0$ such that $\theta(t) \neq 0$), can we find the minimum period τ of θ? In other words, can we find $\tau_0 > 0$ such that $\theta(\tau_0) = 0$ and $\neg(\theta(t) = 0)$ whenever $0 < |t| < \tau_0$? Classically, once we have found $\tau > 0$ as in Theorem 18, we know that τ_0 exists and has the form τ/n for some positive integer n. To make constructive progress, we derive some lemmas, the first three of which are elementary, though both nontrivial and useful.

Lemma 22 *If $y > x > 0$ and $\varepsilon > 0$, then there exists a positive integer N such that either $Nx < y < (N+1)\,x$ or $|y - Nx| < \varepsilon$.*

Proof. Find an integer M such that $Mx < y < (M+2)x$. Then either $|y - (M+1)x| < \varepsilon$, in which case we set $N = M$, or $|y - (M+1)x| > 0$. In the latter case we can find $N \in \{M, M+1\}$ with $Nx < y < (N+1)\,x$. ■

Lemma 23 *If $y > x > 0$, then there exists a positive integer N such that $|Nx - y| < x$.*

Proof. Apply Lemma 22 with $\varepsilon = x$. ■

Lemma 24 *Let $0 < \tau < r < t$. Then there exists a nonnegative integer N such that $0 < t - N\tau < r$.*

Proof. By Lemma 22, there exists a positive integer N such that either $N\tau < t < (N+1)\,\tau$ or

$$|t - N\tau| < \min\{\tau, r - \tau\}\,.$$

In the former case, $0 < t - N\tau < \tau$; in the latter, $-\tau < t - N\tau < r - \tau$, so $0 < t - (N-1)\,\tau < r$. ■

The following result provides a lower bound for the minimal period.

Lemma 25 *Let θ be a continuous homomorphism of* $\mathbf{R}$ *onto a nontrivial compact abelian group G. Then there exists $t_0 > 0$ such that $\theta(t) \neq 0$ whenever $0 < |t| < t_0$.*

Proof. Fix $t_1 > 0$ such that $\theta(t_1) \neq 0$, and choose $t_0 \in (0, t_1)$ such that $\rho(0, \theta(t)) > \frac{1}{2}\rho(0, \theta(t_1))$ whenever $|t - t_1| < t_0$. Let $0 < t < t_0$. By Lemma 24, there exists a nonnegative integer N such that $0 < t_1 - Nt < t_0$. Using the continuity of the mapping $\theta(t) \rightsquigarrow \theta(nt)$, we obtain $\delta > 0$ such that if $\rho(0, \theta(t)) < \delta$, then $\rho(0, \theta(Nt)) < \frac{1}{2}\rho(0, \theta(t_1))$. Since $|Nt - t_1| < t_0$, we have $\rho(0, \theta(Nt)) > \frac{1}{2}\rho(0, \theta(t_1))$; whence $\rho(0, \theta(t)) \geqslant \delta$, $N\theta(t) = \theta(Nt) \neq 0$, and therefore $\theta(t) \neq 0$. If, on the other hand, $-t_0 < t < 0$, then, by the first part of the proof, $\theta(-t) \neq 0$ and therefore again $\theta(t) \neq 0$. ■

Lemma 26 *Under the hypotheses of, and with t_0 as in,* Lemma 25, *if $\tau - t_0 < t < \tau$, then $\theta(t) \neq 0$.*

Proof. For such t we have $-t_0 < t_0 - \tau < 0$, so, by Lemma 25, $\theta(t_0 - \tau) \neq 0$ and therefore $\theta(t_0) \neq \theta(\tau) = 0$. ■

Lemma 27 *In the notation of* Lemma 25*, the restriction of* θ *to the interval* $[0, t_0)$ *is injective.*

Proof. Let t_1, t_2 be distinct points of $[0, t_0)$. Then $0 < |t_1 - t_2| < t_0$, so, by Lemma 25,

$$\theta(t_1) - \theta(t_2) = \theta(t_1 - t_2) \neq 0$$

and therefore $\theta(t_1) \neq \theta(t_2)$. ■

With the lower bound for the minimal period from Lemma 25, we can begin to narrow down the possibilities for the minimal period.

Proposition 28 *Let* θ *be a continuous periodic homomorphism, with period* τ*, of* $\mathbf{R}$ *onto a nontrivial compact abelian group* G*. Then there exists a positive integer* N *such that* $\theta(\tau/n) \neq 0$ *for all* $n > N$*.*

Proof. We need only compute, in turn, $t_0 > 0$ as in Lemma 25, and a positive integer N such that $\tau/N < t_0$. ■

Lemma 29 *Let* θ *be a continuous periodic homomorphism, with period* τ*, of* $\mathbf{R}$ *onto a nontrivial compact abelian group* $\mathbf{G}$*. If* $m, n \in \mathbf{N}$ *are such that*

$$\theta\left(\frac{\gcd(m+n, mn)}{mn}\tau\right) \neq 0,$$

then either $\theta(\tau/n) \neq 0$ *or* $\theta(\tau/m) \neq 0$*.*

Proof. If m, n satisfy the hypothesis, then

$$\begin{aligned}\theta\left(\frac{\tau}{m}\right) + \theta\left(\frac{\tau}{n}\right) &= \theta\left(\frac{(m+n)\tau}{mn}\right) \\ &= \theta\left(k\frac{\gcd(m+n, mn)}{mn}\tau\right) \neq 0,\end{aligned}$$

for some $1 \leqslant k < mn/\gcd(m+n, mn)$. Thus $\theta(\tau/n) \neq \theta(\tau/m)$, and either $\theta(\tau/n) \neq 0$ or $\theta(\tau/m) \neq 0$. ■

Corollary 30 *Let* θ *be a continuous periodic homomorphism of* $\mathbf{R}$ *onto a nontrivial compact abelian group* $\mathbf{G}$ *with period* τ*. If* $M = p_1^{\alpha_1} \cdots p_k^{\alpha_k}$*, where* $p_1, \ldots, p_k$ *are distinct prime numbers, and* $\theta(\tau/M) \neq 0$*, then there exists* $i \in \{1, \ldots, k\}$ *such that* $\theta(\tau/p_i^{\alpha_i}) \neq 0$*.*

Proof. Set $m_1 = p_1^{\alpha_1}$ and $m_2 = p_2^{\alpha_2} \cdots p_k^{\alpha_k}$. Since $p_1, \ldots, p_k$ are distinct prime numbers, $\gcd(m_1 + m_2, m_1 m_2) = 1$; by Lemma 29, there exists $i \in \{1, 2\}$ such that $\theta(\tau/m_i) \neq 0$. Replacing M by M/m_i, the result follows by inducting on k. ■

The next result further narrows our search for the minimal period.

Proposition 31 *Let θ be a continuous periodic homomorphism, with period τ, of $\mathbf{R}$ onto a nontrivial compact abelian group $\mathbf{G}$. Then there exists $M \in \mathbf{N}$ such that if n does not divide M, then $\theta(\tau/n) \neq 0$.*

Proof. Let N be a natural number such that $\theta(\tau/n) \neq 0$ for all $n > N$. Using Lemma 29 repeatedly, construct a subset T of $\{1, \ldots, N\}$ such that for all $n \notin T$, $\theta(\tau/n) \neq 0$ and

$$\forall m, n \in T \left(\frac{mn}{\gcd(m-n, mn)} \in T \right). \tag{6}$$

Let M be the largest element of T. Assume there exists an element of T that does not divide M, and let n be the largest such element of T. Since

$$\frac{Mn}{\gcd(M-n, Mn)} = \frac{\operatorname{lcm}(M-n, Mn)}{M-n} \geqslant \frac{Mn}{M-n} > n,$$

it follows from (6) and the maximality of n that there exists $k \in \mathbf{N}$ such that

$$\frac{Mn}{\gcd(M-n, Mn)} = \frac{M}{k}.$$

But then $\gcd(M-n, Mn) = kn$, so n divides $M-n$ and hence M. This contradiction ensures that every element of T divides M. ∎

With M as in this last proposition, and donning, for the moment, a classical-logical hat, we now see that the minimum period of θ in Theorem 18 is τ/n for some positive integer n dividing M. Doffing that hat again, we provide a Brouwerian example that shows why we cannot produce that n constructively.

Recall the metric abelian group from definitions (2) and (3). We prove that if, under the hypothesis of Theorem 18, we can always find the smallest positive τ such that $\theta_a(t+\tau) = \theta_a(t)$ for all t, then **LLPO** implies **WLPO**.

Assuming **LLPO**, we see from Lemma 16 that for each $a \in \mathbf{R}$, the group G_a is complete; since G_a is totally bounded, it is therefore compact. Moreover, since G_a is periodic, $\theta(0, \infty)$ is open. Suppose that

$$\tau_0 = \inf\{t > 0 : \theta_a(t) = (1, a)\}$$

exists. Either $\tau_0 < 2$ or $\tau_0 > 1$. In the first case, if $a \neq 0$, then $\tau_0 = 2$, a contradiction; whence $a = 0$. In the case $\tau_0 > 1$, we cannot have $(1, -a) = \theta_a(1) = \theta_a(0) = (1, a)$, so $\neg(a = 0)$. Since $a \in \mathbf{R}$ is arbitrary, we have proved that

$$\forall_{x \in \mathbf{R}} \left(x = 0 \vee \neg \left(x = 0 \right) \right),$$

which is constructively equivalent to **WLPO**.

Since **LLPO** is provably weaker than **WLPO** (see [2]), we conclude from the foregoing analysis that under the hypotheses of Theorem 18, we cannot derive the conclusion that there exists a minimum positive value of τ such that $\theta(\tau) = 0$.

A few comments on our results so far regarding the minimal period. Firstly, Lemma 16 and the above argument can be extended to sets of the form

$$G_{a_1,\ldots,a_n} = \left\{ \left(e^{p_1 \cdots p_n \pi i t}, a_1 e^{p_1 \pi i t}, \ldots, a_n e^{p_n \pi i t} \right) : t \in \mathbf{R} \right\}, \tag{7}$$

where $a_1, \ldots, a_n \in \mathbf{R}$ and $p_1, p_2, \ldots, p_n$ are distinct prime numbers, to show that Proposition 31 is the best possible in our constructive framework. Secondly, from a constructive point of view, it is perhaps more natural to insist that $\theta(t) \neq 0$ for all $t \in (0, \tau_0)$ when defining the minimal period τ_0. Throughout the rest of the chapter we adopt this definition; we denote this stronger minimal period of a homomorphism by $\tau_{\min}$. With this definition the above argument would show that the existence of a minimal period implies that **LLPO** and **LPO** are equivalent.

Our next task is to find conditions under which the minimal period exists[9]; we consider the simplest case: namely, when G is contained in $\mathbf{R}^2$. Before we can state our main result (Theorem 32) we need a few definitions. Let G be the image of $\mathbf{R}$ under a differentiable map. Then G is said to satisfy the *twin tangent ball condition* if G is contained in $\mathbf{R}^2$ and there exists $\nu > 0$ such that for each $x \in G$ there exist points a_x, b_x on opposite sides of the tangent line at x such that

$$\overline{B}\left(a_x, \nu\right) \cap G = \{x\} = \overline{B}\left(b_x, \nu\right) \cap G.$$

If G is a *Jordan curve*—that is, if G is the range of a uniformly continuous mapping $f_G : S^1 \to \mathbf{R}^2$ with uniformly continuous inverse[10]—then a_x, b_x belong to separate components[11] of $\mathbf{R}^2 - G$.

Theorem 32 *Let θ be a differentiable homomorphism from $\mathbf{R}$ onto a non-trivial compact abelian group G that satisfies the twin tangent ball condition. If θ is periodic, then θ has a minimal period.*

The key to proving Theorem 32 is the idea that, if we know the minimal period of θ, then we can associate G with a Jordan curve. Now considering a Jordan curve we can use the following (rather involved) lemma from [13]; our statement differs slightly from the one in that reference.

Lemma 33 *Let J be a differentiable Jordan curve that satisfies the twin tangent ball condition. Then there exists $\beta > 0$ such that if $r \in (0, \beta], w \in \mathbf{R}^2, 0 \leqslant t_1 < t < t_2 \leqslant 2\pi$, and $\left\| f_J\left(e^{it_k}\right) - w \right\| \leqslant r \quad (k = 1, 2)$, then $\left\| f_J\left(e^{it}\right) - w \right\| < r$.*

[9]The main result of the remainder of this section and the results proof have been greatly generalised and greatly simplified respectively; see [21].

[10]We associate every Jordan curve with such a function.

[11]The Jordan curve theorem ensures that $\mathbf{R}^2 - G$ has two components, the inside of G and the outside of G [10].

For the remainder of this chapter we assume that θ and G satisfy the hypothesis of Theorem 32. Let $\tau > 0$ and $M \in \mathbf{N}$ be such that $\theta(\tau) = 0$ and $\theta(\tau/n) \neq 0$ for all $n \in \mathbf{N}$ not dividing M. Let $d_1, \ldots, d_l$ be the proper divisors of M, and let p be the smallest prime not in $\{d_1, \ldots, d_l\}$. Define

$$\begin{aligned} m_k &= \sup\left\{\rho\left(\theta(t), \theta\left(t + \frac{\tau}{d_k}\right)\right) : t \in [0, \tau]\right\}, \text{ and} \\ \delta_k &= \inf\left\{\rho\left(\theta(t), \theta\left(t + \frac{\tau}{pd_k}\right)\right) : t \in [0, \tau]\right\} > 0, \end{aligned}$$

for each $k \in \{1, \ldots, l\}$.

Lemma 34 *Let $\tau > 0$ and $M \in \mathbf{N}$ be such that $\theta(\tau) = 0$ and $\theta(\tau/n) \neq 0$ whenever n does not divide M. Then either there exists $k \in \{1, \ldots, l\}$ such that $m_k > 0$, or else it is impossible that $m_k > 0$ for each k $(1 \leqslant k \leqslant l)$.*

Proof. If there exists k such that $m_k > 0$, then we are done; so we may assume that $m_k < \delta_k$ for each k. Assume then that $m_k > 0$ for each k; then $\tau = \tau_{\min}$. We show that the inverse of θ restricted to $\theta[0, \tau)$—for convenience we denote this by θ^{-1}—is uniformly continuous, in which case we can associate G with a Jordan curve. We first establish continuity at 0, for which it suffices to show that

$$\rho\left(0, \{\theta(t) : t \in [r, \tau - r]\}\right) > 0$$

for each $r \in (0, \tau/2)$. Construct a continuous injective homomorphism $\phi : \mathbf{R} \to \mathbf{R}^3$ with the following properties:

- If $t < 0$, then $\phi(t) = (\theta(r), t)$;
- If $0 \leqslant t < \tau - r$, then $\phi(t) = (\theta(t), 0)$;
- If $t \geqslant \tau - r$, then $\phi(t) = (\theta(\tau - r), t - (\tau - r))$.

With addition defined by $\phi(s) + \phi(t) = \phi(s + t)$ and with the metric induced from $\mathbf{R}^3$, $G' = \phi(\mathbf{R})$ is a metric abelian group. To see that G' is complete, let $(y_n)_{n\geqslant 1}$ be a Cauchy sequence in G'. For each n write $y_n = (x_n, t_n)$, where $x_n \in G$ and $t_n \in \mathbf{R}$. Then $(x_n)_{n\geqslant 1}$ and $(t_n)_{n\geqslant 1}$ are Cauchy sequences in the complete spaces G and $\mathbf{R}$. Compute $x \in G$ and $t \in \mathbf{R}$ such that $x_n \to x$ and $t_n \to t$ as $n \to \infty$. Then $y = (x, t) \in G'$, by the continuity of θ, and $y_n \to y$ as $n \to \infty$. Noting that $\{\theta(t) : |t| > 1\}$ is locally totally bounded and therefore located, we see from Corollary 18 of [14] that

$$\{\theta(t) : |t| > r\},$$

and hence

$$\{\theta(t) : t \in [r, \tau - r]\},$$

is bounded away from 0 for each $r > 0$. To show that θ^{-1} is uniformly continuous, fix $\varepsilon > 0$ and let $\delta > 0$ be such that if $\rho(0, x) < \delta$, then $\theta^{-1}(x) < \varepsilon$. Using

[6] (page 400, Proposition (1.2)), compute $\delta' > 0$ such that for all $x, y \in G$ if $\rho(x,y) < \delta'$, then $\rho(0, x-y) < \delta$. So if $\rho(x,y) < \delta'$, then $\rho(0, x-y) < \delta$ and

$$\left|\theta^{-1}(x) - \theta^{-1}(y)\right| = \left|\theta^{-1}(x-y)\right| < \varepsilon.$$

Associating G with a Jordan curve and letting β be as in Lemma 33, we may assume, without loss of generality, that $\|0 - \theta\left(\tau/d_1\right)\| < \beta/2$. Let

$$w = \frac{\left(0 + \theta\left(\tau/d_1\right)\right)}{2}$$

and

$$r = \|0 - w\| = \|\theta\left(\tau/2\right) - w\| < \beta.$$

It follows from Lemma 33 that $\|\theta\left(\tau/pd_1\right) - w\| < r$, so

$$\begin{aligned} \delta_1 &\leqslant \|\theta\left(\tau/pd_1\right) - 0\| \\ &\leqslant \|\theta\left(\tau/pd_1\right) - w\| + \|w - 0\| \\ &< 2r \\ &= \rho\left(0, \theta\left(\tau/d_1\right)\right) \quad \leqslant \quad m_1, \end{aligned}$$

a contradiction. Hence it is impossible for $m_k > 0$ for each k $(1 \leqslant k \leqslant n)$. ■

Lemma 35 *Let $\tau > 0$ and $M \in \mathbf{N}$ be such that $\theta(\tau) = 0$, and $\theta(\tau/m) \neq 0$ whenever n does not divide M. Then either $m_k > 0$ for each k $(1 \leqslant k \leqslant l)$ or else it is impossible that $m_k > 0$ for all k.*

Proof. We proceed by induction on l. The case $l = 1$ follows directly from Lemma 34. Assume the result holds for the case $l = n-1$ and consider the case $l = n$. Applying Lemma 34, we see that either it is impossible that $m_k > 0$ for all k $(1 \leqslant k \leqslant n)$, in which case we are done, or else there exists $k \in \{1, \ldots, n\}$ such that $m_k > 0$. In this latter case, replacing M by M/p_k, we obtain the result from our induction hypothesis. ■

We now have the **proof of Theorem 32**:

Proof. Let $\tau > 0$ and $M \in \mathbf{N}$ be such that $\theta(\tau) = 0$ and $\theta(\tau/n) \neq 0$ whenever n does not divide M. We again proceed by induction on the number of prime factors of M. If M is prime, then by Lemma 34, either $m_M > 0$, in which case $\tau = \tau_{\min}$, or else $m_M = 0$ and $\tau/M = \tau_{\min}$. Now assume that the result holds when M has $r-1$ prime factors, and consider the case $M = p_1^{\alpha_1} \cdots p_l^{\alpha_l}$, where $p_1, \ldots, p_l$ are distinct prime numbers, $\alpha_j \in \mathbf{N}^+$ for each j, and $\alpha_1 + \cdots + \alpha_l = N$. Let

$$\begin{aligned} D &= \{d \neq M : d \text{ divides } M\} = \{d_1, d_2, \ldots, d_i\}, \text{ and} \\ A_k &= \{j : d_k \text{ divides } d_j\}. \end{aligned}$$

Applying Lemma 35 to each of the pairs $\tau/d_k, M/d_k$ $(1 \leqslant k \leqslant i)$, in each case under the assumption that $\theta(\tau/d_k) = 0$, we can divide $\{1,\ldots,i\}$ into disjoint sets P, Q such that

$$\begin{aligned} k \in P \quad &\Rightarrow \quad (\theta(\tau/d_k) = 0 \Rightarrow (\forall\ k \in A_k\ m_k > 0)) \\ k \in Q \quad &\Rightarrow \quad (\theta(\tau/d_k) = 0 \Rightarrow \neg(\forall\ k \in A_k\ m_k > 0)). \end{aligned}$$

If $k \in Q$ and $\tau/d_k = \tau_{\min}$, then $\theta(\tau/d_k) = 0$ and $m_k > 0$ for all $k \in A_k$—a contradiction. Hence if $k \in Q$, then $\neg(\tau/d_k = \tau_{\min})$. On the other hand, if $k \in P$, it follows that

$$\theta(\tau/d_k) = 0 \Rightarrow \theta(\tau/M) \neq 0.$$

Let $q \in \mathbf{Q}^+$ be such that

$$\theta(\tau/d_k) = 0 \Rightarrow \theta(\tau/M) > q$$

for all $k \in P$. Then either $\theta(\tau/M) < q$, in which case $\neg(\tau/d_k = \tau_{\min})$ for each $k \in P$ and hence for all k $(1 \leqslant k \leqslant i)$, or else $\theta(\tau/M) > 0$. In the first case, $\tau/M = \tau_{\min}$. In the second case, by Corollary 30, there exists $j \in \{1,\ldots,l\}$ such that $\theta\left(\tau/p_j^{\alpha_j}\right) \neq 0$. If n does not divide M/p_j, then either n does not divide M and so $\theta(\tau/n) \neq 0$, or $p_j^{\alpha_j}$ divides n. In this latter case, compute $\delta > 0$ such that if $\rho(0,\theta(t)) < \delta$, then $\rho\left(0,\theta\left((n/p_j^{\alpha_j})t\right)\right) < \rho\left(0,\theta(\tau/p_j^{\alpha_j})\right)$. Then $\rho(0,\theta(\tau/n)) \geqslant \delta$. Applying our induction hypothesis to M/p_j completes the proof. ∎

Throughout the proof of Theorem 32 the only use we make of the hypothesis that G is contained in $\mathbf{R}^2$ occurs in Lemma 33. Hence if we could extend this result to higher dimensions, then we could extend Theorem 32 similarly. Such an extension, however, is likely to be non-trivial since the proof of Lemma 33 in [13] is surprisingly difficult and makes use of the Jordan curve theorem, which does not extend to higher dimensions.

Chapter 4

Continuous isomorphisms of R onto a complete group

In this section we consider Theorem 2 within **BISH**; we give a constructive version of this theorem under a locatedness condition. A special case occurs when G satisfies a certain local path-connectedness condition at 0. It is also shown that Theorem 2 holds in **BISH** + **BD-N**.

4.1 Proving noncompactness

Our constructive version of Theorem 2 is

Theorem 36 *Let θ be a continuous, one-one homomorphism of* $\mathbf{R}$ *onto a complete (metric) abelian group G, and suppose that*

$$S_1 = \{\theta(t) : |t| \geqslant 1\}.$$

is weakly located at 0*. Then G is noncompact.*

Note that Theorems 18 and 36 are classically equivalent to each other. For if the former and the hypotheses of the latter hold, then θ cannot be one-one; whereas if the latter and the hypotheses of the former hold, then $\ker\theta = \{0\}$, so G cannot be compact. However, viewed constructively, these two theorems about group homomorphisms are quite distinct.

If θ is a homomorphism of the additive abelian group $\mathbf{R}$ into a group G, then for each $r > 0$ we write

$$S_r = \{\theta(t) \in G : |t| \geqslant r\}. \tag{8}$$

Lemma 37 *Let θ be a continuous homomorphism of* $\mathbf{R}$ *onto a metric abelian group G such that $\rho(0, S_1) = 0$. Then $\rho(0, S_r) = 0$ for each $r > 0$.*

Proof. First fix $r \geqslant 1$ and $\varepsilon > 0$. Pick a positive integer $N > r$. There exists $\delta \in (0, \varepsilon)$ such that if $x \in G$ and $\rho(0, x) < \delta$, then $\rho(0, Nx) < \varepsilon$. Pick $t \in S_1$ such that $\rho(0, \theta(t)) < \delta$. Then $|Nt| \geqslant N > r$ and $\rho(0, \theta(Nt)) = \rho(0, N\theta(t)) < \varepsilon$. Since $\varepsilon > 0$ is arbitrary, the desired conclusion follows. ■

Lemma 38 *Let θ be a continuous homomorphism of* $\mathbf{R}$ *onto a metric abelian group G such that $\rho(0, S_1) > 0$. Then $\rho(0, S_r) > 0$ for each $r > 0$.*

Proof. Compute a positive integer N such that $Nr > 1$. There exists $\delta > 0$ such that if $x \in G$ and $\rho(0, x) < \delta$, then $\rho(0, Nx) < \rho(0, S_1)$. For each t with $|t| \geqslant r$ we have $\theta(Nt) \in S_1$, so $\rho(0, N\theta(t)) = \rho(0, \theta(Nt)) \geqslant \rho(0, S_1)$; whence $\rho(0, \theta(t)) \geqslant \delta$. ∎

Corollary 39 *Let θ be a continuous homomorphism of* $\mathbf{R}$ *onto a metric abelian group G such that $\rho(0, S_1) > 0$. Then θ is injective.*

The final part of the next proof follows the standard classical route described in section 1.4.

Proposition 40 *If θ is a continuous, one-one homomorphism of* $\mathbf{R}$ *onto a complete metric abelian group G, then $\neg(\rho(0, S_1) = 0)$.*

Proof. Assume that $\rho(0, S_1) = 0$. Then we can construct a sequence $(t_n)_{n \geqslant 1}$ of real numbers such that $|t_1| < |t_2| < |t_3| < \cdots \to \infty$ and $\theta(t_n) \to 0$ as $n \to \infty$; whence, by Lemma 4, **LPO** holds. On the other hand, by Corollary 9, for each positive integer n, $\theta[-n, n]$ is (compact and) closed in G. Since G is complete and

$$G = \bigcup_{n \geqslant 1} \theta[-n, n],$$

Corollary 14 shows that there exist a positive integer N, $t_0 \in \mathbf{R}$, and $r > 0$ such that $B(\theta(t_0), r) \subset \theta[-N, N]$. In view of our initial assumption and Lemma 37, there exists t such that $|t| > N$ and $\rho(\theta(t), \theta(t_0)) < r$. Hence there exists $t' \in \theta[-N, N]$ such that $\theta(t) = \theta(t')$. This is absurd, since $t' \neq t$ and θ is one-one. ∎

Corollary 41 *Let θ be a continuous, one-one homomorphism of* $\mathbf{R}$ *onto a complete metric abelian group G, such that S_1 is weakly located at 0. Then $\rho(0, S_r) > 0$ for each $r > 0$.*

Proof. By Proposition 3, θ is injective. It follows from Lemma 5 that either $\rho(0, S_1) > 0$ or $\rho(0, S_1) = 0$. The latter is ruled out by Proposition 40. It remains to invoke Lemma 38. ∎

We now turn our attention to the inverse of θ, which provides the key to proving Theorem 36.

Proposition 42 *Let θ be a one-one homomorphism of* $\mathbf{R}$ *onto a metric abelian group G, and suppose that θ^{-1} is continuous at some point of G. Then θ^{-1} is continuous at each point of G.*

Proof. Suppose that θ^{-1} is continuous at the point $\theta(t_0) \in G$. Given $\varepsilon > 0$, choose $\delta_1 > 0$ such that if $t \in \mathbf{R}$ and $\rho(\theta(t), \theta(t_0)) < \delta_1$, then $|t - t_0| < \varepsilon$. Consider any $t \in \mathbf{R}$. Since the mapping

$$x \rightsquigarrow x + \theta(t_0 - t)$$

is continuous at $\theta(t)$, there exists $\delta > 0$ such that if $s \in \mathbf{R}$ and $\rho(\theta(s), \theta(t)) < \delta$, then

$$\rho(\theta(s + t_0 - t), \theta(t_0)) = \rho(\theta(s) + \theta(t_0 - t), \theta(t) + \theta(t_0 - t)) < \delta_1$$

and therefore

$$|s - t| = |(s + t_0 - t) - t_0| < \varepsilon.$$

Hence θ^{-1} is continuous at t. ■

Proposition 43 *Under the hypotheses of* Corollary *41, the homomorphism* $\theta^{-1} : G \to \mathbf{R}$ *is pointwise continuous on* G*, and uniformly continuous on each compact subset of* G*.*

Proof. Fix $\varepsilon > 0$. By Corollary 41, there exists $\delta > 0$ such that $\rho(0, S_\varepsilon) > \delta$. If $\rho(0, \theta(t)) < \delta$, then $|t| \leqslant \varepsilon$. Hence θ^{-1} is continuous at 0 and therefore, by Proposition 42, at each point of G.

Now let K be a compact subset of G. By the first part of the proof, there exists $\alpha > 0$ such that if $x \in G$ and $\rho(0, x) < \alpha$, then $\left|\theta^{-1}(x)\right| < \varepsilon$. On the other hand, since the mapping $(x, y) \rightsquigarrow y - x$ is uniformly continuous on $K \times K$, there exists $\delta > 0$ such that if $x, x', y, y' \in K$ and $\rho((x, y), (x', y')) < \delta$, then $\rho(x - y, x' - y') < \alpha$. In particular, if $x, y \in K$ and $\rho(x, y) < \delta$, then $\rho(0, x - y) < \alpha$, so

$$\left|\theta^{-1}(x) - \theta^{-1}(y)\right| = \left|\theta^{-1}(x - y)\right| < \varepsilon.$$

This establishes the uniform continuity of θ^{-1} on K. ■

We can now complete the **Proof of Theorem 36**:

Proof. Under the hypotheses of that theorem, let K be a compact subset of G. By Proposition 43, θ^{-1} is uniformly continuous on K; so $\theta^{-1}(K)$ is totally bounded. Let $t \in \mathbf{R} - \theta^{-1}(K)$. It follows from the continuity of θ^{-1} at $\theta(t)$ that $\theta(t) \in G - K$. ■

Corollary 44 *Let θ be a continuous, one-one homomorphism of* $\mathbf{R}$ *onto a locally compact (metric) abelian group G with S_1 weakly located at* 0*. Then G is unbounded.*

Proof. Given $R > 0$, let K be a compact subset of G containing $B(0, R)$. Then, by Theorem 36, there exists $x \in G - K \subset G - B(0, R)$. ■

The final proposition of this section shows, in a strong way, that if $r > 0$ and $\theta[0, r]$ has inhabited metric complement, then no point of $(0, r]$ is a period of G. It is easy to do this if we are allowed to use Markov's principle. For, given $t \in (0, r]$ and supposing that $\theta(t) = 0$, Proposition 20 shows that $\theta[0, t)$ is dense in G, which is absurd; whence, by Markov's principle, $\theta(t) \neq 0$. Without Markov's principle we must work harder.

Proposition 45 *Let $r, t_0 > 0$, and suppose that $\rho(\theta(t_0), \theta[0,r]) > 0$. Then $\theta(t) \neq 0$ for each $t \in (0,r]$.*

Proof. By continuity, $0 < r < t_0$ and there exists $r' \in (r, t_0)$ such that $\rho(\theta(t_0), \theta[0,r']) > 0$. Replacing r by r', if necessary, it suffices to show that $\theta(t) \neq 0$ for all $t \in (0,r)$. Fix $t \in (0,r)$ and let

$$\varepsilon = \rho(\theta(t_0), \theta[0,r]).$$

Using Lemma 24, compute a positive integer N such that $t_0 - Nt \in (0,r)$. By the continuity of the mapping $(x,y) \rightsquigarrow y - x$, there exists $\delta > 0$ such that, for all x in G, if $\rho(0,x) < \delta$, then $\rho(\theta(t), \theta(t) - x) < \varepsilon$. In turn, by the continuity of the mapping $\theta(t) \rightsquigarrow \theta(Nt)$ at $(0,0)$, there exists $\delta_1 > 0$ such that if $t \in [0,r]$ and $\rho(0,\theta(t)) < \delta_1$, then $\rho(0,\theta(Nt)) < \delta$. Suppose that $\rho(0,\theta(t)) < \delta_1$. Then $\rho(0,\theta(Nt)) < \delta$; so, by our choice of N and δ,

$$\begin{aligned}
\rho(\theta(t_0), \theta[0,r]) &\leqslant \rho(\theta(t_0), \theta(t_0 - Nt)) \\
&= \rho(\theta(t_0), \theta(t_0) - \theta(Nt)) \\
&< \varepsilon = \rho(\theta(t_0), \theta[0,r]),
\end{aligned}$$

which is absurd. Hence $\theta(t) \neq 0$. ■

4.2 Comments on the hypothesis

We now comment on the hypothesis that S_1 is weakly located at 0. That hypothesis is used to prove, in Corollary 41, that $\rho(0, S_r) > 0$ for each $r > 0$, and hence, in Proposition 43, that θ^{-1} is pointwise continuous on G (from which the uniform continuity of θ^{-1} on compact sets follows). It is easy to see that the pointwise continuity of θ^{-1} at 0 is *equivalent* to the weak locatedness of S_1 at 0. Without the latter hypothesis, but still making use of Proposition 43, we can prove the following two propositions.

Proposition 46 *Let θ be a continuous isomorphism of* **R** *onto a complete abelian group G. Then θ^{-1} is sequentially continuous on G.*

Proof. It clearly suffices to prove that θ^{-1} is sequentially continuous at 0. Accordingly, let $\theta(t_n) \to 0$ in G, and let $\varepsilon > 0$. By Ishihara's tricks, either $|t_n| < \varepsilon$ eventually or else $|t_n| > \varepsilon/2$ infinitely often. In the latter case, by Lemma 6.6.9 of [12], **LPO** holds; so S_1, being separable, is located in G. It follows from Proposition 43 that θ^{-1} is pointwise continuous at 0, which is absurd, since $|t_n| > \varepsilon/2$ infinitely often. Thus, in fact, $|t_n| < \varepsilon$ eventually. Since $\varepsilon > 0$ is arbitrary, we conclude that θ^{-1} is sequentially continuous at 0. ■

Proposition 47 *Let θ be a continuous, one-one homomorphism of* **R** *onto a complete (metric) abelian group (G, ρ), and let $r > 0$. Then $\theta[-r,r]$ is compact, and the restriction of θ^{-1} to $\theta[-r,r]$ is uniformly continuous.*

Proof. Let $0 < \varepsilon < r$. Since θ is uniformly continuous on the totally bounded intervals $(\varepsilon, r]$ and $[r, \varepsilon)$, the set

$$S = \{\theta(t) : \varepsilon < |t| \leqslant r\},$$

is totally bounded. Construct a strictly decreasing sequence $(\delta_n)_{n\geqslant 1}$ of positive numbers converging to 0 such that for each n, if $x \in G$ and $\rho(0, x) < \delta_n$, then $\rho(0, nx) < 2^{-n}$. Construct an increasing binary sequence $(\lambda_n)_{n\geqslant 1}$ such that

$$\lambda_n = 0 \Rightarrow \rho(0, S) < \delta_n,$$
$$\lambda_n = 1 \Rightarrow \rho(0, S) > \delta_{n+1}.$$

If $\lambda_n = 0$, choose t_n such that $\varepsilon < |t_n| \leqslant r$ and $\rho(0, \theta(t_n)) < \delta_n$; setting $x_n = \theta(nt_n)$, we have $\rho(0, x_n) < 2^{-n}$. If $\lambda_n = 1 - \lambda_{n-1}$, set $t_m = t_{n-1}$ and $x_m = x_{m-1}$ for all $m \geqslant n$. Then $(x_n)_{n\geqslant 1}$ is a Cauchy sequence in G and so converges to a limit $x_\infty \in G$. Let $\tau = \theta^{-1}(x_\infty)$, and compute a positive integer $N > |\tau| / \varepsilon$. Suppose that $\lambda_N = 0$. If $\lambda_m = 1 - \lambda_{m-1}$ for some $m > N$, then $\varepsilon < |t_{m-1}| \leqslant r$ and $\theta(\tau) = \theta((m-1)t_{m-1})$. Since θ is one-one, $\tau = (m-1)\, t_{m-1}$ and therefore

$$|\tau| = (m-1)\, |t_{m-1}| \geqslant (m-1)\, \varepsilon \geqslant N\varepsilon,$$

a contradiction. Hence $\lambda_n = 0$ for all $n \geqslant N$ and therefore for all n. Thus $|nt_n| > n\varepsilon \to \infty$, but $\theta(nt_n) \to 0$. This contradicts Proposition 46. Hence the case $\lambda_N = 0$ is ruled out, and we have $\lambda_N = 1$. Thus $\rho(0, S) > 0$. It follows that if $|t| \leqslant r$ and $\rho(0, \theta(t)) < \delta_{N+1}$, then $\theta(t) \notin S$ and therefore $|t| \leqslant \varepsilon$.

We can now compute $\alpha > 0$ such that if $t \in [-2r, 2r]$ and $\rho(0, \theta(t)) < \alpha$, then $|t| < \varepsilon$. Corollary 8 shows that $K = \theta[-r, r]$ is compact; so the mapping $(x, y) \rightsquigarrow y - x$ is uniformly continuous on $K \times K$. Pick $\delta > 0$ such that if $x, x', y, y' \in K$ and $\rho((x, y), (x', y')) < \delta$, then $\rho(y - x, y' - x') < \alpha$. If $t, t' \in [-r, r]$ and $\rho(\theta(t), \theta(t')) < \delta$, then

$$\rho((\theta(t), \theta(t)), (\theta(t'), \theta(t))) < \delta$$

and so

$$\rho(0, \theta(t - t')) = \rho(0, \theta(t) - \theta(t')) < \alpha.$$

Since $|t - t'| \leqslant 2r$, it follows that $|t - t'| < \varepsilon$. Since $\varepsilon > 0$ is arbitrary, the proof is complete. ■

In [23] (Theorem 4), Ishihara showed that the statement

> Every sequentially continuous mapping of a separable metric space into a metric space is pointwise continuous

is equivalent, in **BISH**, to **BD-N**. Since **BD-N** holds in the classical, intuitionistic, and recursive models of **BISH** (again see [23]), it follows from Proposition 46 and the remark preceding it that the statement

> (*) For each continuous isomorphism of $\mathbf{R}$ onto a complete abelian group, the image of $\{t \in \mathbf{R} : |t| \geqslant 1\}$ is weakly located at 0

holds in each of these models. This precludes our obtaining a Brouwerian or a recursive counterexample to (*). Also, in these models, Theorem 2 holds without additional hypothesis.

Propositions 46 and 47 enable us to show that the weak locatedness hypothesis in Theorem 36 can be derived if we assume a form of local path connectedness for the group G. Note that, unlike the weak locatedness of S_1 at 0, this is an intrinsic property of the group G, independent of the isomorphism $\theta : \mathbf{R} \to G$.

Recall that a metric space X is *path connected* if for all $x, x' \in X$ there exists a (uniformly continuous) path $\gamma : [0,1] \to X$ with $\gamma(0) = x$ and $\gamma(1) = x'$.

Lemma 48 *Let X be a path connected metric space, and f a sequentially continuous mapping of X into $\mathbf{R}$. Let $a, b \in X$ and $t \in \mathbf{R}$ satisfy $f(a) < t < f(b)$. Then for each $\varepsilon > 0$, there exists $x \in X$ such that $|f(x) - t| < \varepsilon$.*

Proof. There exists a uniformly continuous mapping $\gamma : [0,1] \to X$ such that $\gamma(0) = a$ and $\gamma(1) = b$. Then $f \circ \gamma$ is a sequentially continuous mapping of $[0,1]$ into $\mathbf{R}$ such that $f \circ \gamma(0) < t < f \circ \gamma(1)$. It follows from a well-known result (see Exercise 17 on page 58 of [12]) that for each $\varepsilon > 0$, there exists $\zeta \in [0,1]$ such that $|f(\gamma(\zeta)) - t| < \varepsilon$. ■

Let a be a point of a metric space X. We say that X is *ball-locally path connected at a* if there exists $\tau > 0$ such that $\overline{B}(a, r)$ is path connected for all but countably many $r \in (0, \tau)$.

Proposition 49 *Let θ be a continuous, one-one homomorphism of $\mathbf{R}$ onto a complete (metric) abelian group G that is ball-locally connected at 0. Then θ^{-1} is pointwise continuous.*

Proof. By Proposition 42, it is enough to prove that θ^{-1} is continuous at 0. Since θ is injective, we can compute $r > 0$ such that

$$0 < r < \frac{1}{2} \min \{\rho(0, \theta(1)), \rho(0, \theta(-1))\}$$

and $\overline{B}(0, r)$ is path connected. We show that

$$\theta^{-1}\left(\overline{B}(0, r)\right) \subset [-1, 1]. \tag{9}$$

Note that $\rho\left(\theta(1), \overline{B}(0, r)\right) > r$ and $\rho\left(\theta(-1), \overline{B}(0, r)\right) > r$. Now compute $\alpha > 0$ such that if $t, t' \in [-1, 1]$ and $|t - t'| < \alpha$, then $\rho(\theta(t), \theta(t')) < r$. Given $t \in \theta^{-1}\left(\overline{B}(0, r)\right)$, suppose that $|t| > 1$. If $t > 1$, then since $0 < 1 < t$ and θ^{-1}

is sequentially continuous on G, it follows from Lemma 48 there exists t' such that $1-\alpha < t' < 1$ and $\theta(t') \in \overline{B}\,(0,r)$. By our choice of α,

$$\rho\left(\theta(1), \overline{B}\,(0,r)\right) \leqslant \rho\left(\theta(1), \theta(t')\right) < r,$$

a contradiction. The other possibility, $t < -1$, is ruled out similarly. We conclude that $t \in [-1,1]$. Hence (9) holds.

Given $\varepsilon > 0$ and using Lemma 47, now choose $\delta > 0$ such that if $t, t' \in [-1,1]$ and $\rho\left(\theta(t), \theta(t')\right) < \delta$, then $|t-t'| < \varepsilon$. If $\rho\left(0,x\right) < \min\left\{\delta, r\right\}$ in G, then $x \in \overline{B}(0,r) \subset \theta\left[-1,1\right]$, so $\left|\theta^{-1}(x)\right| < \varepsilon$. Since $\varepsilon > 0$ is arbitrary, we conclude that θ^{-1} is pointwise continuous at 0. Reference to Proposition 42 completes the proof. ■

Corollary 50 *Let θ be a continuous, one-one homomorphism of* $\mathbf{R}$ *onto a complete (metric) abelian group (G,ρ) that is ball-locally path connected at* 0*. Then $\rho(0,S_1) > 0$ and S_1 is weakly located at* 0*.*

Proof. By Proposition 49, there exists $\delta > 0$ such that if $\rho\left(0,\theta(t)\right) < \delta$, then $|t| < 1$. Thus if $|t| \geqslant 1$, then $\rho\left(0,\theta(t)\right) \geqslant \delta$, from which the desired conclusions follow. ■

We now return to the condition "$\rho\left(0,S_1\right) > 0$" that, as we pointed out earlier, is vital for the proof of our main theorem. First, we give a result that is worth including for the sake of completeness of exposition.

If, under the hypotheses of Theorem 36, the group G is locally compact, then, as a consequence of Corollary 41 and our next proposition, we can prove that S_1 is located.

Proposition 51 *Let θ be a continuous homomorphism of* $\mathbf{R}$ *onto a locally compact abelian group G such that $\rho\left(0,S_1\right) > 0$, where S_r is defined at* (8)*. Then S_r is located in G for each $r > 0$.*

Proof. Fix r, R such that

$$B = \overline{B}\,(0, R+1) \cap G$$

is compact and contains the (totally) bounded set $\theta\left[-r-1, r+1\right]$. Let $0 < \varepsilon < 1$, and choose $\delta \in (0, 1/2)$ such that for all $t, t' \in [-r-1, r+1]$, if $|t-t'| \leqslant \delta$, then $\rho\left(\theta(t), \theta(t')\right) < \varepsilon/2$. In view of Lemma 38, there exists $c > 0$ such that $\rho(0, S_\delta) > c$. Now, since G is locally compact, the mapping $(t,x) \rightsquigarrow \theta(t) + x$ is uniformly continuous on the compact subset $[-r,r] \times B$ of $\mathbf{R} \times G$, so we can choose $\alpha \in (0, \varepsilon/2)$ such that if $y, z \in B$ and $\rho(y,z) < \alpha$, then

$$\rho\left(y + \theta(t), z + \theta(t)\right) < c \quad \left(t \in [-r,r]\right).$$

Let $t \in [-r, r]$, and let $|t'| \geqslant r + \delta$ be such that $\theta(t') \in B$. Suppose that $\rho(\theta(t), \theta(t')) < \alpha$. Then $-t \in [-r, r]$, so

$$\rho(0, \theta(t' - t)) = \rho(\theta(t - t)), \theta(t' - t)) < c,$$

which contradicts our choice of c, since $|t' - t| \geqslant \delta$. Thus

$$\rho(y, \theta[-r, r]) \geqslant \alpha \qquad (y \in S_{r+\delta} \cap B). \tag{10}$$

Construct a finite $\alpha/2$-approximation $\{\xi_1, \ldots, \xi_m\}$ to B, and write $\{1, \ldots, m\}$ as a union of sets P, Q such that

$$\begin{aligned} i \in P &\Rightarrow \rho(\xi_i, \theta[-r, r]) > 0, \\ i \in Q &\Rightarrow \rho(\xi_i, \theta[-r, r]) < \frac{\alpha}{2}. \end{aligned}$$

Consider t such that $\theta(t) \in S_r \cap B$. There exists $y \in S_{r+\delta} \cap B$ such that $\rho(\theta(t), y) < \varepsilon/2$. Indeed, either $|t| > r + \delta$, when we take $y = \theta(t)$, or else $|t| < r + 1 - \delta$; in the latter case, since $\theta[-r - 1, r + 1] \subset B$, it follows from our choice of δ that setting $y = \theta\left(t + \frac{t}{|t|}\delta\right)$ fulfils our requirements. Pick i such that $\rho(y, \xi_i) < \alpha/2$. If $i \in Q$, then

$$\begin{aligned} \rho(y, \theta[-r, r]) &\leqslant \rho(y, \xi_i) + \rho(\xi_i, \theta[-r, r]) \\ &< \frac{\alpha}{2} + \frac{\alpha}{2} = \alpha, \end{aligned}$$

which contradicts (10). Hence $i \notin Q$ and therefore $i \in P$. It follows that $\{\xi_i : i \in P\}$ is a finitely enumerable ε-approximation to $S_r \cap B$. Since $\varepsilon > 0$ is arbitrary, it follows that $S_r \cap B$ is totally bounded. Hence S_r is locally totally bounded and therefore (by Proposition 2.2.18 of [12]) located. ■

In the notation of Theorem 36, can we prove, without assuming local compactness of our group G, that if it is noncompact, then S_1 is located? (We have just shown that we can when G is locally compact and $\rho(0, S_1) > 0$.) The answer is 'no': the statement

> If θ is a continuous, one-one homomorphism of $\mathbf{R}$ onto a complete abelian group G, such that $\rho(0, S_1) > 0$, then S_1 is located in G

implies **LPO**. To prove this, consider any binary sequence $(a_n)_{n \geqslant 1}$ with at most one term equal to 1. Construct a continuous injection $\theta : \mathbf{R} \to \mathbf{C}$ with the following properties:

- if either $t \leqslant 1$ or $a_n = 0$ for all $n \leqslant t$, then $\theta(t) = (2t, 0)$;
- if $a_n = 1$ and $n < t \leqslant n + 1$, then $\theta(t) = (2n, t - n)$;
- if $a_n = 1$ and $n + 1 < t$, then $\theta(t) = (4n + 2 - 2t, 1)$.

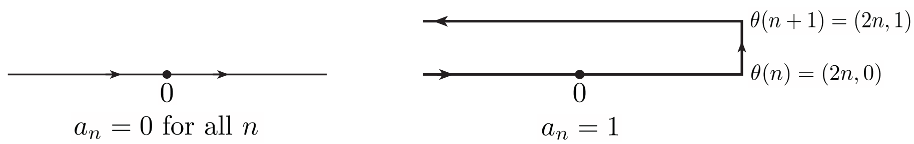

Figure 4: What does the image of $\mathbf{R}$ under θ look like?

Then θ is a continuous isomorphism between $\mathbf{R}$ and the group $G = \theta(\mathbf{R})$, taken with the Euclidean metric and the addition defined by

$$\theta(t) + \theta(t') = \theta(t + t') \qquad (t, t' \in \mathbf{R}).$$

To prove that G is complete, we first observe

(a) that if $a_N = 1$, then G is the closure in $\mathbf{C}$ of the set

$$((-\infty, N) \times \{0\}) \cup \{(N, y) : 0 \leqslant y \leqslant 1\} \cup ((-\infty, N] \times \{1\})$$

and so is complete; and

(b) that if $\operatorname{Im}\theta(t) < 1$, then $\operatorname{Re}\theta(t) \geqslant 2t - 2$.

Let $(t_n)_{n\geqslant 1}$ be any sequence in $\mathbf{R}$ such that $(\theta(t_n))_{n\geqslant 1}$ is a Cauchy sequence in G. Then $z = \lim_{n\to\infty} \theta(t_n)$ exists in $\mathbf{C}$. Either $\operatorname{Im} z > 0$ or $\operatorname{Im} z < 1$. In the first case, computing in turn positive integers N, m such that $\operatorname{Im}\theta(t_N) > 0$ and $t_N < m$, we see that $a_n = 1$ for some $n \leqslant m$; so, by observation (a), G is complete. In the case $\operatorname{Im} z < 1$, compute a positive integer $M > \frac{1}{2}\operatorname{Re} z$. There exists κ such that $\operatorname{Re}\theta(t_k) < 2M$ and $\operatorname{Im}\theta(t_k) < 1$ for all $k \geqslant \kappa$. For such a value of k, suppose that $t_k > M+1$. Then $\operatorname{Re}\theta(t_k) < 2t_k - 2$; so, by observation (b), $\operatorname{Im}\theta(t) \geqslant 1$, a contradiction. Hence $t_k \leqslant M + 1$ for all $k \geqslant \kappa$. If $a_n = 0$ for all $n \leqslant M + 1$, then

$$z = \lim_{k\to\infty, k\geqslant\kappa} \theta(t_k) = \lim_{k\to\infty} (2t_k, 0),$$

so

$$t = \frac{1}{2}\operatorname{Re} z = \lim_{k\to\infty} t_k \leqslant M + 1$$

and $z = (2t, 0) = \theta(t)$. If, on the other hand, $a_n = 1$ for some $n \leqslant M + 1$, then G is complete. This completes the proof of completeness for G.

It is clear that $\rho(0, S_1) \geqslant 1$. Now suppose that

$$s = \inf\{\rho(\theta(0), \theta(t)) : |t| \geqslant 1\}$$

exists. Either $s > 1$ and therefore $a_n = 0$ for all n, or else $s < 2$. In the latter case we can choose t with $|t| \geqslant 1$ and $\rho(\theta(0), \theta(t)) < 2$. If $t < -1$, then $\rho(\theta(0), \theta(t)) = 2|t| < 2$, a contradiction; so $t \geqslant 1$. Compute a positive integer

$\nu > t$. If $a_n = 0$ for all $n \leqslant \nu$, then $\rho(0, \theta(t)) = 2t \geqslant 2$, which is absurd. Hence $a_n = 1$ for some $n \leqslant \nu$.

Note that in this example, the group G is ball-locally path connected at 0: for $0 < r < 1$, the set

$$\overline{B}(0, r) = [-r, r] \times \{0\}$$

is actually connected. However, $\overline{B}(0, 1)$ is not path connected if there exists n with $a_n = 1$. Moreover, if G is locally compact, then either $a_n = 0$ for all n or else there exists n such that $a_n = 1$. For there exists a compact subset K of G such that

$$\forall_{t \in \mathbf{R}} \left(\rho\left(\theta(0), \theta(t)\right) < 2 \Rightarrow \theta(t) \in K\right).$$

Pick $t_1, \ldots, t_m$ such that $\{\theta(t_1), \ldots, \theta(t_m)\}$ is a 1/2-approximation to K. Either $\operatorname{Im}\theta(t_j) < 1$ for all j, in which case $a_n = 0$ for all n; or else there exists j such that $\operatorname{Im}\theta(t_j) > 0$, when there exists n with $a_n = 1$.

Chapter 5

Open problems

In this thesis we have presented a constructive consideration of the classical theorem

Theorem 1 *Let θ be a continuous homomorphism of $\mathbf{R}$ onto a compact (metric) abelian group G. Then there exists $\tau > 0$ such that $\theta(\tau) = 0$.*

and its contrapositive

Theorem 2 *Let θ be a continuous, one-one homomorphism of $\mathbf{R}$ onto a complete (metric) abelian group G. Then G is noncompact.*

within the framework of Bishop's constructive mathematics. However, our constructive versions of Theorems 1 and 2 each involve extra hypothesis. The natural question to ask is then: Can we do better, or are Theorems 1 and 2 fundamentally non-constructive?

In particular, is the assumption that $\theta(0,\infty)$ is open in G necessary in Theorem 18? The next corollary, which follows directly from Theorem 18 and the discussion following it, helps clarify this question.

Corollary 52 *Let θ be a continuous homomorphism of $\mathbf{R}$ onto a compact (metric) abelian group G. Then the following are equivalent.*

(1) *θ is periodic.*

(2) *$\theta(0,\infty)$ is open.*

(3) *There exists a sequence $(R_n)_{n\geqslant 1}$ of real numbers strictly increasing to infinity such that $\theta\left[-R_n, R_n\right]$ is weakly bilocated at 0 for each n.*

(4) *There exists $t > 0$ such that $G = \theta[0,t]$.*

Proposition 12 shows that **LPO** implies condition *(3)* of the above corollary; whence Theorem 1 holds in **BISH** + **LPO**. Moreover, in any Brouwerian counterexample to Theorem 1 we cannot have $\theta(0,\infty)$ open. Such a group and homomorphism, suggested by Hannes Diener as a potential Brouwerian counterexample to Theorem 1, is given by $\theta : \mathbf{R} \to G$, where $\theta(t) = ae^{iat}$ for some $a \in \mathbf{R}$, and $G = \theta(\mathbf{R})$ with the metric induced from $\mathbf{C}$. Then $\theta(0,\infty)$

is open in G if and only if $a = 0$ or $a \neq 0$. To see this, let $\delta > 0$ be such that $B(\theta(1), \delta) \subset \theta(0, \infty)$. If $|a| > 0$ we are done, so we may assume that $|a| < \delta/2$. Then $\theta(-1) \in B(\theta(1), \delta) \subset \theta(0, \infty)$, so there exists $t > 0$ such that $\theta(-1) = \theta(t)$; that is, such that $ae^{-ia} = ae^{iat}$. Either $|a(t+1)| > 0$, in which case $a \neq 0$, or $|a(t+1)| < 2\pi$. In the latter case if $a \neq 0$, then $e^{ia(t+1)} \neq 1$—a contradiction. Hence $\neg(a \neq 0)$ and therefore $a = 0$.

However, Hannes showed, with the next result, that this is not the group we are after.

Proposition 53 *Let $a \in \mathbf{R}$ and define $\theta : \mathbf{R} \to G$ by $\theta(t) = ae^{iat}$. Then $G = \theta(\mathbf{R})$, with the metric induced from $\mathbf{C}$, is complete if and only if $a = 0$ or $a \neq 0$.*

Proof. Without loss of generality we may assume that $|a| < 1$. Construct an increasing binary sequence $(\lambda_n)_{n \geqslant 1}$ such that

$$\begin{aligned} \lambda_n = 0 \quad &\Rightarrow \quad |a| < \frac{1}{n}; \\ \lambda_n = 1 \quad &\Rightarrow \quad |a| > \frac{1}{n+1}. \end{aligned}$$

If $\lambda_n = 0$, set $x_n = \theta(0) = a$; if $\lambda_n = 1 - \lambda_{n-1}$, set $x_k = \theta(\pi/|a|) = -a$ for all $k \geqslant n$. Then $(x_n)_{n \geqslant 1}$ is a Cauchy sequence in G. If G is complete, then there exists $t_\infty \in \mathbf{R}$ such that $x_n \to \theta(t_\infty)$ as $n \to \infty$. Pick an integer N such that $t_\infty < \pi N$. Either $\lambda_N = 1$ and $a \neq 0$, or $\lambda_N = 0$ and $|a| < 1/N$. In the latter case, if $\lambda_m = 1$ for some $m > N$, then $t_\infty = \pi/|a| > \pi N$—a contradiction. Hence $\lambda_n = 0$ for all n, so $a = 0$. ∎

The problem of finding the minimal period of a homomorphism is a little more complex. We have shown that, under the hypothesis of Theorem 18, we cannot hope in general to find the minimal period of θ, but that $\tau_{\min}$ exists when θ is differentiable and G satisfies the twin tangent ball condition. So, can we find more general conditions under which the minimal period exists? Our first task might be to generalise Lemma 33, and hence Theorem 32, to n-dimensions. With this in mind, we define an *n-dimensional closed curve* to be a one-one uniformly continuous mapping $f : S^1 \to \mathbf{R}^n$ with a uniformly continuous inverse. A curve G in $\mathbf{R}^n$ is said to satisfy the *encircled tangent ball condition* if there exists $\nu > 0$ such that for each $x \in G$ and all $y \in (f'(x))^\perp \cap \partial B(x, \nu)$ we have

$$\overline{B}(y, \nu) \cap G = \{x\}.$$

We might then hope to prove

Conjecture 54 *Let G be a differentiable n-dimensional closed curve that satisfies the encircled tangent ball condition. Then there exists $\beta > 0$ such that if $r \in (0, \beta], w \in \mathbf{R}^n, 0 \leqslant t_1 < t < t_2 \leqslant 2\pi$, and $\left\| f\left(e^{it_k}\right) - w \right\| \leqslant r \quad (k = 1, 2)$, then $\left\| f\left(e^{it}\right) - w \right\| < r$.*

in which case we would have

Conjecture 55 [12] *Let θ be a differentiable homomorphism from* $\mathbf{R}$ *onto a nontrivial compact abelian group G that is contained in* $\mathbf{R}^n$ *and satisfies the encircled tangent ball condition. If θ is periodic, then θ has a minimal period.*

Our Brouwerian example establishing that we cannot always construct the minimal period of a periodic homomorphism involved a subset, G_a, of the 4-dimensional space $S^1 \times \mathbf{C}$; this can, however, easily be recast as a subset of the 3-dimensional space $S^1 \times \mathbf{R}$. So, can we perhaps construct the minimal period of θ whenever G is contained in a 2-dimensional space? With the abelian groups $G_{a_1,\ldots,a_n}$, introduced in (7), in mind we might suggest the next conjecture. Note that $2^m = |\mathcal{P}(\{p_1, \ldots, p_m\})|$ is the number of distinct positive divisors of $M = p_1 \cdots p_m$.

Conjecture 56 *Let θ be a continuous periodic homomorphism of* $\mathbf{R}$ *onto a compact (metric) abelian group G. If G is contained in* $\mathbf{R}^n$*, then there exists a subset $\{\tau_1, \ldots, \tau_k\}$ of* $\mathbf{R}$ *such that $k \leqslant 2^{n-2}$ and $\theta(\tau_{\max}) = 0$, where $\tau_{\max} = \max\{\tau_1, \ldots, \tau_k\}$.*

With Theorem 2 we are slightly closer to having the complete picture. We have shown that **BD-N** is sufficient to prove Theorem 2 and, consequently, that this Theorem holds in **INT** and **RUSS** in addition to **CLASS**. The major remaining questions are then: Can we prove Theorem 2 within **BISH**? and if not, then does Theorem 2 imply **BD-N**? If this latter case holds, then can we establish Theorem 2 in the special case that our abelian group is locally compact?

Theorem 36 shows that in any Brouwerian counterexample to Theorem 2 we cannot have S_1 weakly located at 0. Forgetting for the moment that Theorem 2 holds in **BISH** + **BD-N**, we might try defining $G = \theta(\mathbf{R})$ where $\theta(t) = at$ for some $a \in \mathbf{R}$ such that $\neg(a = 0)$. However, not surprisingly, Proposition 3 shows that the completeness of this set is equivalent to Markov's principle[13]. Another possible approach would be to let $(a_n)_{n \geqslant 1}$ be a binary sequence with at most one nonzero term and to define a continuous function θ from $\mathbf{R}$ into $\mathbf{R}^2$ such that:

- if $t \leqslant 1$, then $\theta(t) = (t, 0)$;
- if $t > 1$ and $a_n = 0$ for all $n \leqslant t$, then $\theta(t) = (1, t-1)$;
- if $a_n = 1$ and $n < t \leqslant n+1$, then

$$\theta(t) = \left(1 - n - kt, \frac{1 + n - n^2}{n} t + n^2 - 2\right);$$

[12]This conjecture, and the periodicity of the homomorphisms in question, is proved in [21].
[13]In fact, the metric space $\mathbf{R}a$ is complete if and only if $a = 0$ or $a \neq 0$.

▶ if $a_n = 1$ and $n+1 < t$, then $\theta(t) = (0, 1/n + t - n + 1)$.

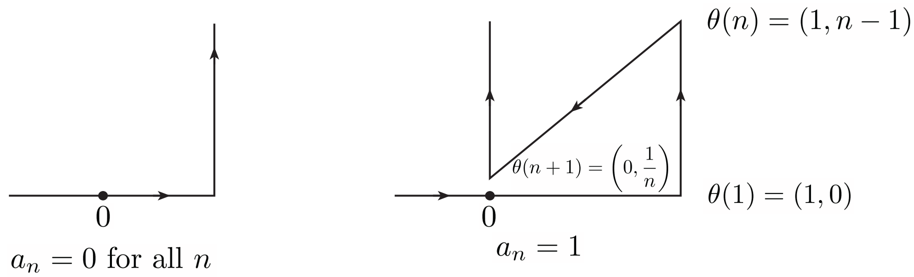

Figure 5: What does the image of $\mathbf{R}$ under θ look like?

If $a_n = 1$ for some n, then $\rho(0, S_1) = 1/n$, and if $a_n = 0$ for all n, then $\rho(0, S_1) = 1$. However, once again we cannot show the completeness of $G = \theta(\mathbf{R})$: assume G is complete. If $a_k = 0$ for all $k \leqslant n$, set $x_n = 0$. If $a_n = 1$, then set $x_k = \theta(n+1)$ for each $k \geqslant n$. Then $(x_n)_{n \geqslant 1}$ is a Cauchy, and hence convergent, sequence in G. Let $t_\infty \in \mathbf{R}$ be such that $x_n \to \theta(t_\infty)$ as $n \to \infty$, and pick $N > 0$ such that $N > t_\infty$. Suppose that $a_N = 0$ and $a_m = 1 - a_{m-1}$ for some $m > N$. Then $\theta(t_\infty) = \theta(m+1)$, so $t_\infty = m + 1 > N$. This contradiction ensures that either $a_N = 1$ or $a_n = 0$ for each n. Regardless of this, it is easy to see that G is unbounded (and hence noncompact): given $r > 0$, either $\rho(0, \theta(r)) > r$ or $\rho(0, \theta(2r+1) > r$.

As we have seen, the difficulty in finding Brouwerian counterexamples to Theorems 1 and 2 arises in finding abelian groups which are 'nice' enough to be complete, but which are not periodic or noncompact, respectively. In fact, our one successful Brouwerian counterexample concerning the generalisation of **COP** required us to assume a non-constructive principle (**LLPO**) in order to establish the completeness of the abelian group under consideration. Given that being complete appears to impose so much structure on our complete abelian groups, it seems likely that Theorems 1 and 2 admit constructive proofs.

On the other hand, it is interesting to note that there is some similarity between the extra hypothesis assumed in Theorems 18 and 36. In Theorem 18 we are concerned with showing that G is 'small', in that it is periodic and therefore the image, under a uniformly continuous mapping, of a compact space. Here we assume that $\theta(0, \infty)$ is open, in which case $\theta(-\infty, 0)$ is also open, so S_1 is open. In Theorem 36 we show that G is 'large', in that it is noncompact; here we assume that S_1 is weakly located at 0. By Corollary 41, this is equivalent to $\rho(0, S_1) > 0$, which is in turn equivalent[14] to $\theta(-r, r) = G - S_r$ (Proposition 10) being open for each $r > 0$. So to show that G is 'small' we have assumed that the image under θ of a large part of $\mathbf{R}$ is open, and to show that G is 'large' we

[14]This follows from Lemma 38 and [6] (Page 400, Proposition (1.2)).

have assumed that the image under θ of a small part of $\mathbf{R}$ is open. We have also shown that the extra locatedness hypothesis of Theorem 36 holds when G is ball locally path connected at 0; it seems likely that this condition is also sufficient to establish that $\theta(0, \infty)$ is open when θ is a continuous homomorphism onto a compact abelian group.

A final comment: our investigation of continuous group homomorphisms arose from Arno Bergers asking whether the Baire category theorem was essential for the proof of the classical 'compact orbits are periodic' theorem, whose abstraction is studied in this thesis. We still do not know the answer to Berger's question, but it is interesting that the proof of our direct constructive analogue, Theorem 18, of the (abstracted) periodicity theorem uses the 'dense open sets' version of Baire's theorem, whereas our proof of the contrapositive Theorem 36 uses the constructively inequivalent 'union of closed sets' version presented as Theorem 13.

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